IEOR 160 Midterm Question Solutions

Question 1 Solution

a) Let x denote the number of computers Eric will buy. The nonlinear programming formulation is

$$\begin{aligned} \max \quad & -2x^2 + 300x \\ \text{s.t.} \quad & 50 \leq x \leq 100 \end{aligned}$$

b) The KKT condition of the problem is

$$\begin{gathered} 50 \leq x \leq 100 \\ -4x + 300 - \lambda_1 + \lambda_2 = 0 \\ \lambda_1(100 - x) = 0, \;\; \lambda_1 \geq 0 \\ \lambda_2(x - 50) = 0, \;\; \lambda_2 \geq 0 \end{gathered}$$

c) The solution for the KKT condition in b) is $x = 75$, $\lambda_1 = 0$, $\lambda_2 = 0$.

d) Since the Hessian of the objective function in a) is $[-4]$ which is negative definite, the objective function is concave, and since the constraints are linear and the problem is a maximization problem, the KKT conditions give optimal solution to the problem.

Question 2 Solution

a) The problem can be formulated as

$$\begin{aligned} \max \quad & -y^2 + axy + by + c \\ \text{s.t.} \quad & x + y = 10 \end{aligned}$$

b) The Hessian of the objective function in a) is $\begin{pmatrix} 0 & a \\ a & -2 \end{pmatrix}$. For the objective function to be concave, its Hessian must negative semidifinite. Therefore, if $a = 0$, b and c free, the objective function will be concave.

c) The Lagrangian function is

$$L(x, \lambda) = -y^2 + axy + by + c + \lambda(100 - x - y)$$

d) Take the derivative of the Lagrangian rpt x, y and λ, we get

$$\begin{gathered} ay - \lambda = 0 \\ -2y + ax + b - \lambda = 0 \\ x + y = 10. \end{gathered}$$

Thus, $x = \frac{10a-b+20}{2a+2}$, $y = \frac{10a+b}{2a+2}$, and $\lambda = \frac{10a^2+ab}{2a+2}$.

e) When $a = 0$, the point in d) is optimal. When $a = 0$, $x = \frac{-b+20}{2}$, $y = \frac{b}{2}$ and $\lambda = 0$.

Question 3 Solution

1) False. Consider the following NLP:

$$\max \quad -x^4$$

It is easy to see that $x = 0$ is the optimal solution for the NLP, yet the Hessian of it is $[0]$, which is not negative definite.

2) False. For example, $f(x) = x$ is a concave function, yet it has no local minimum or local maximum.

3) True. Since $x_1^2 + x_2^2$ is a convex function, $(0.5x_1^* + 0.5y_1^*)^2 + (0.5y_2^* + 0.5y_2^*)^2 \leq (0.5(x_1^*)^2 + 0.5(x_2^*)^2) + (0.5(y_1^*)^2 + 0.5(y_2^*)^2) \leq 10$, and similarly the second constraint is satisfied as well at $0.5x^* + 0.5y^*$. Thus, $0.5x^* + 0.5y^*$ is feasible. Since $f(x_1, x_2)$ is a convex function, $f(0.5x^* + 0.5y^*) \leq 0.5f(x^*) + 0.5f(y^*) = f(x^*) = f(y^*)$, therefore, $0.5x^* + 0.5y^*$ is also a minimum point.