ME 109 Exam 2 Solutions

Put a box around your answer

(If you refer to an eq. or fig. in the text or notes give the page number)

(30) 1. A pure aluminum sphere, 10 cm (.1 m) in diameter, is initially at 320 K. The sphere is suddenly exposed to air at 300K and at the same time a uniform volumetric heat source (1000 W/m^3) is suddenly activated. Write the governing unsteady energy equation that applies and is needed for the determination of the temperature of the sphere.

Do NOT solve but provide all the necessary quantities that would be needed to solve for the temperature (or refer to the equation you would use to evaluate these quantities).

Repeat: Do NOT solve.

Important properties of air ($T_f \approx 300\ K$):

$$\Pr_{air} = 0.707$$
$$\alpha_{air} = 22.5 \cdot 10^{-6}\ \tfrac{m^2}{s}$$
$$\beta = 1/T = 1/300 = 0.0033\ K^{-1}$$
$$k_{air} = 26.3 \cdot 10^{-3}\ \tfrac{W}{m-K}$$
$$\nu_{air} = 15.9 \cdot 10^{-6}\ \tfrac{m^2}{s}$$

Important property of aluminum ($T \approx 300\ K$):

$$k_{Al} = 237\ \tfrac{W}{m-K}$$
$$\rho_{Al} = 2700\ \tfrac{kg}{m^3}$$
$$c_p = 900 \tfrac{J}{kg-K}$$

This is a lumped capacitance problem. Lumped capacitance is a scenario in which the energy entering or leaving a body is considered to be rate limited by convection at the surface. That is, the resistance associated with conduction to the surface is much less than the resistance associated with convection to the environment and thus the body's temperature deviates minimally and, further can be approximated as a constant over the body.

In this problem the convection is caused by free convection because buoyancy-driven forces are driving the convection. There are two non-dimensional groups that relate buoyant forces to viscous forces, the *Grashof number*, Gr_D and the *Rayleigh number*, Ra_D. Here the subscript '*D*' is used as the characteristic length for cylinders and spheres, whereas 'L' would be used for a plane wall. Depending on the correlation, the *Nusselt* correlation may be dependent on either number. For this particular correlation, the Rayleigh number is required. Ra_D is found as the product of Gr_D and Pr:

$$Ra_D = Gr_D \Pr = \frac{g\beta(T_s - T_\infty)D^3}{\nu^2} \Pr = 91400\,(T_s - T_\infty)$$

but note that this value contains the surface temperature which varies with time, $T_s(t)$ so $Ra_D(t)$. Free convection for spheres is given by equation (9.35) on page 583:

$$\overline{Nu_D}(t) = 2 + \frac{0.589\ Ra_D^{1/4}(t)}{\left(1 + (0.469/\Pr)^{9/16}\right)^{4/9}} = 2 + 7.90\,(T_s(t) - T_\infty)^{1/4}$$

$\overline{Nu_D}(t)$ can then be converted into an average heat transfer coefficient, $\bar{h}(t)$ as follows

$$\bar{h}(t) = \frac{\overline{Nu_D}(t)\,k_{air}}{D} = \frac{\overline{Nu_D}(t)\,k_{air}}{D} = 0.53 + 2.08\,(T_s(t) - T_\infty)^{1/4}$$

In order to proceed we must first check that the *Biot number* (Bi) meets the criterion for applying a lumped capacitance analysis, equation (5.10) on page 261:

$$Bi = \frac{\bar{h}\,L_c}{k_{Al}} \le 0.1,\ Bi = \frac{\bar{h}\ r}{3k_{Al}} = 0.0021$$

since $\bar{h}$ will decrease as T_s approaches T_∞, $\bar{h}(t = 0)$ is used to find the largest $Bi(t)$. The lumped capacitance criterion is met and lumped capacitance can be assumed.

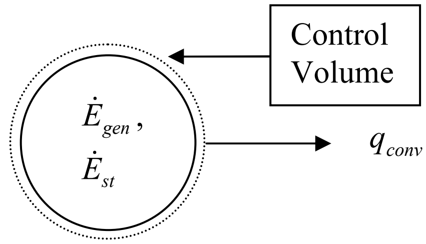
Since the temperature is assumed to be constant throughout the entire body, the total stored energy, $E_{st}(t)$ in the system can be thought of as a function of only time, t:

$$E_{st}(t) = m\,c_p\,T = \rho \forall\, c_p\,T(t)$$

The energy balance within the control volume then contains three terms: the time-rate of energy stored, $\dot{E}_{st}(t)$, the energy generated, $\dot{E}_{gen}$ and free convection, $q_{conv}(t)$. The energy rate balance at the control volume is shown in the figure below:

Energy rate balance:
accumulation = in – out

$$\dot{E}_{st}(t) = \dot{E}_{gen} - q_{conv}(t)$$



The terms in the energy balance can be rewritten as:

$$\dot{E}_{st}(t) = \rho\,(\tfrac{4}{3}\pi\, r^3)\, c_p\, \frac{dT}{dt}$$

$$\dot{E}_{gen} = (\tfrac{4}{3}\pi\, r^3)\, \dot{q}_{gen}$$

$$q_{conv}(t) = 4\pi\, r^2\, \overline{h}(t)\, \left(T(t) - T_\infty\right)$$

Expanding the energy balance gives **(DIFFERENTIAL EQUATION)**:

$$\frac{dT}{dt} + \frac{3\,\overline{h}}{\rho\, c_p\, r}\left(T(t) - T_\infty\right) - \frac{\dot{q}}{\rho\, c_p} = 0$$

$$\Rightarrow \quad \frac{dT}{dt} + 1.95\cdot 10^{-4}\left(T - 300\right)^{5/4} + 4.94\cdot 10^{-5}\left(T - 300\right) - 4.12\cdot 10^{-4} = 0$$

with the following initial condition **(NECESSARY IC)**:

$$T(t=0) = 320\ K$$

ME 109 Exam 2 Solutions

Do NOT solve by the finite difference method.
(40) 2. Consider a pure aluminum plate that extends from $x = 0$ to $x = L = 1$ m. The initial temperature is 300 K. At time $t = 0$ the surface at $x = L$ is changed and kept at 330 K while the surface at $x = 0$ insulated.

What is the temperature at $x = 0$ at the times
a) $t = 80$ *seconds*
b) $t = 8000$ *seconds*
Repeat Do NOT solve by the finite difference method.

SOLUTION

Relevant material properties:

$$\alpha_{Al}(300\ K) = 97.1 \cdot 10^{-6}\ \tfrac{m^2}{s} \qquad (Table\ A.1,\ page\ 929)$$
$$k_{Al}(300\ K) = 237\ \tfrac{W}{m-K}$$

Other important parameters:

$$t^* \equiv Fo \equiv \frac{\alpha t}{L^2} \qquad (dimensionless\ time)$$
$$Bi \equiv \frac{hL}{k_{Al}} = \frac{R_{cond}}{R_{conv}} \qquad (dimensionless\ surface\ cond / conv\ ratio)$$
$$x^* \equiv \frac{x}{L} \qquad (dimensionless\ position)$$
$$\theta^* \equiv \frac{\theta}{\theta_i} = \frac{T(x^*, t^*) - T_\infty}{T_i - T_\infty} \qquad (dimensionless\ temperature)$$

Important calculations:

$time = 80\ s$:

$Fo = 0.0078$

$Bi \to \infty$

$x^* = 0$

$time = 8000\ s$:

$Fo = 0.78$

$Bi \to \infty$

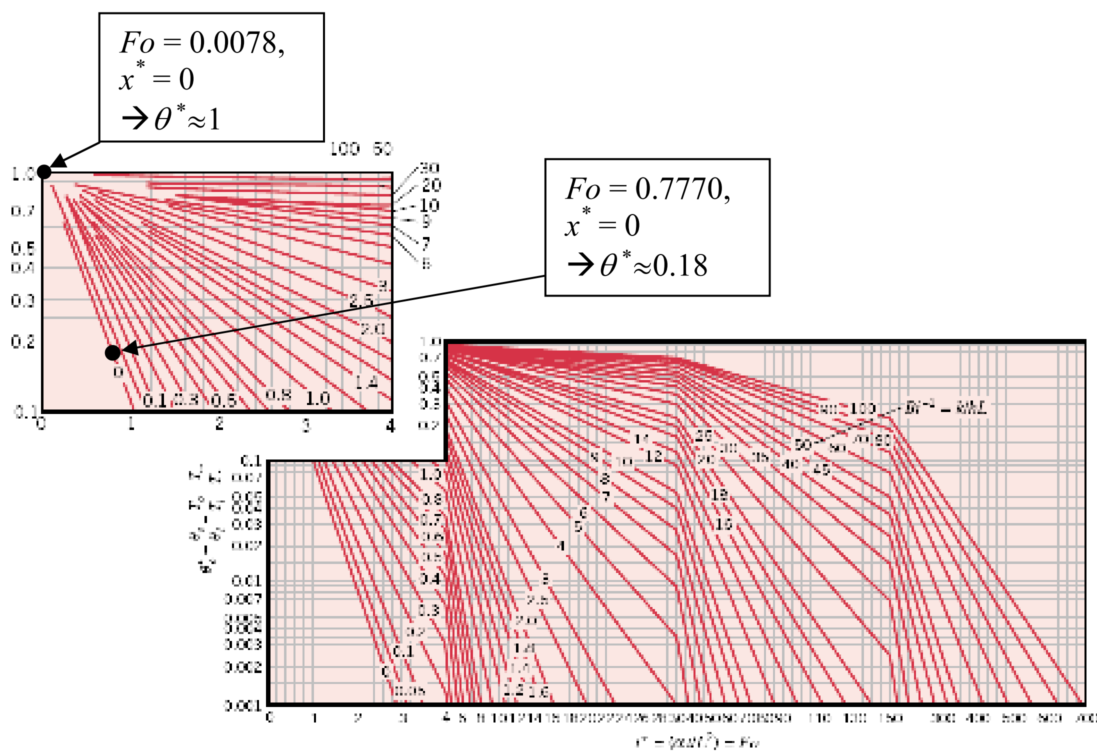
$x^* = 0$

Note that in both cases the *Biot number* (*Bi*) goes to ∞ or $Bi^{-1} = 0$ (value important in figure). This is because the boundary condition at $x = L = 1$ m ($x^*=1$) is

$$-k_{Al}\left.\frac{\partial T}{\partial x}\right|_{x=L} = h\left[T(L,t) - T_\infty\right] \qquad or \qquad \frac{\partial \theta^*}{\partial x^*} = -Bi\,\theta^*\left(1,t^*\right)$$

This is a convection boundary condition but what we want is a dirichlet or temperature-setting boundary condition ($T(x{=}L,t) = 330$ K). The way to proceed is to let $T_\infty = 330\ K$; but we then need $T(x{=}L,t) = T_\infty$. To ensure that this condition is met, the resulting surface heat transfer condition can then be thought of in two separate ways. (1) The convective resistance must be so small that there is negligible temperature drop between $T(x{=}L,t)$ and T_∞ (analogous to no resistance in a series circuit causes no voltage drop), $R_{conv} \sim 1/h = 0$, $h \rightarrow \infty$, Bi→∞. (2) It can be assumed that the surface flux is finite, $q \sim h^*(T(x{=}L,t) - T_\infty)$ in order to force $T(x{=}L,t) = T_\infty$ (or $T(x{=}L,t) - T_\infty = 0$) then $h \rightarrow \infty$, Bi→∞.

Useful figure (*figure 5S.1*):



There are two possible solution techniques exist: (1) using textbook supplements *5.S1* or (2) using the textbook analysis conducted in pages 271-274.

Option (*1*):

See figure above. Following from the calculations of *Fo* and *Bi*, the value of θ^* can be found from extrapolation of the figure (t=80 s, $\theta^*\approx 1$; t=8000 s, $\theta^*\approx 0.18$). As a result of the definition for θ^* above, the temperature $T(x^*, t^*)$ can be found from:

$$T(x^*, t^*) = \theta^*(T_i - T_\infty) + T_\infty$$

Then the temperatures are $T(x=0, t=80\ s) \approx 300\ K$ and $T(x=0, t=8000\ s) \approx 325\ K$.

Option (*2*):

The textbook (page 273) states that a one term approximation of equation (5.39a) can be made for $Fo > 0.2$. However, the $t = 80\ s$ ($Fo = 0.0078$) doesn't meet this requirement. Thus more terms are required to approximate $\theta_0^*(t = 80\ s)$. Then multiple terms must be used to evaluate textbook equation (5.39a)

$$\theta^* = \sum_{n=1}^{\infty} C_n \exp\left(-\zeta_n^2\ Fo\right)\cos\left(\zeta_n\ x^*\right)$$

The book gives the first eigenvalue (ζ_1) on page 274 and the first four terms in Appendix *B.3* (page 962). These values are solutions to textbook equation (5.39c):

$$\zeta_n \tan \zeta_n = Bi = \infty$$

Knowing that ζ_n must be finite, leads us to the equation

$$\tan \zeta_n = \infty$$

for which the eigenvalues can be expressed as

$$\zeta_n = \pi/2 + (n-1)\pi$$

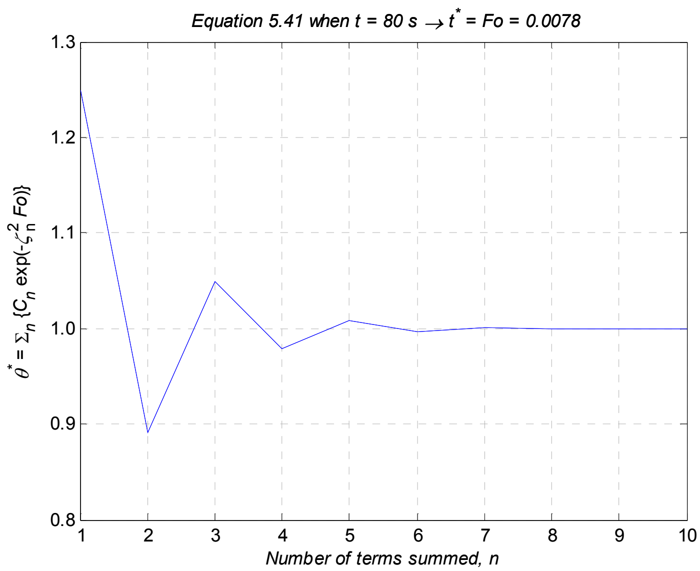
Equation (5.39b) can then be used to find the coefficients, C_n:

$$C_n = \frac{4 \sin(\zeta_n)}{2\zeta_n + \sin(2\zeta_n)}$$

The table below shows the values of ζ_i, C_i, θ_i and θ^*.

i	ζ_i	C_i	θ_i	$\theta^* = \Sigma\theta_i$
1	1.5708	1.2732	1.2490	1.2490
2	4.7124	-0.4244	-0.3569	0.8921
3	7.8540	0.2546	0.1574	1.0494
4	10.9956	-0.1819	-0.0708	0.9786
5	14.1372	0.1415	0.0298	1.0084
6	17.2788	-0.1157	-0.0113	0.9971
7	20.4204	0.0979	0.0038	1.0009
8	23.5619	-0.0849	-0.0011	0.9998
9	26.7035	0.0749	0.0003	1.0001
10	29.8451	-0.0670	-0.0001	1.0000

The following figure plots the convergence of these terms:



From the plot above the value of $\theta_0^*(t = 80\ s)$ converges to 1. So $T(x{=}0,t{=}80\ s) \approx 300\ K$.

For $T(x^*{=}0,t{=}8000$ s or $Fo = 0.78)$, the condition ($Fo > 0.2$) is met to apply the one term approximation in equation (5.41) which considers the temperature as the midplane ($x^*{=}0$)

$$\theta_0^* = C_1 \exp\left(-\zeta_1^2\ Fo\right)$$

using the values in Table 5.1 on page 274 gives

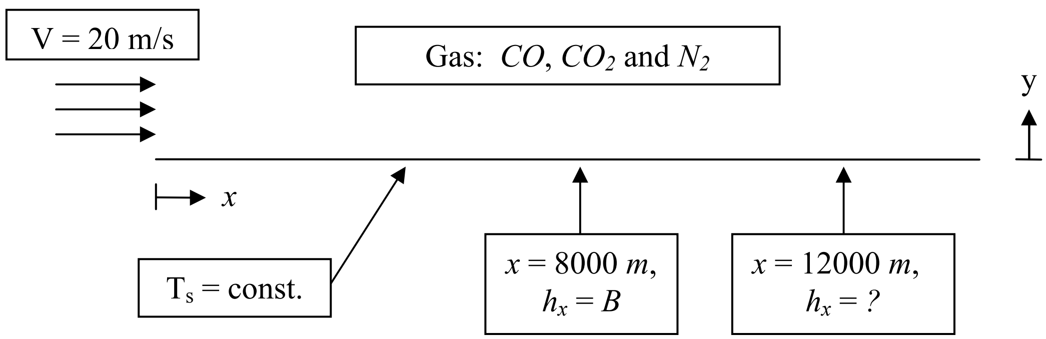
$$\theta_0^*\left(Fo = 0.78\right) = 1.2733\ \exp\left(-\left(\tfrac{\pi}{2}\right)^2\ 0.78\right) = 0.1858$$

Then from the definition of θ^*, $T(x{=}0,t{=}8000\ s) \approx 324.4\ K$.

ME 109 Exam 2 Solutions

(40) 3. Consider boundary layer flow past a constant temperature flat plate that is aligned with the flow. At a location 8 km (= 8000 m) from the leading edge the heat transfer coefficient, h, is measured and we designate the value as B (w/m^2K).

The free stream velocity is 20 m/s. The gas is a mixture of gases (say of nitrogen, oxygen and carbon monoxide). What is the value of the heat transfer coefficient at the location 12 km from the leading edge.



This problem requires the relationship, $h_x(x)$. This is a scaling problem. Several approaches could be used, two are explained below: use of the thermal boundary layer (physical approach) and use of a Nusselt relationship (correlation approach).

Begin by determining flow regime from Reynolds number with a flow velocity of 20 m/s, a position of 8000 m and a kinematic viscosity of ~10^{-5} m^2/s (at 300 K):

$$\mathrm{Re}_x = \frac{Vx}{\nu} = \frac{20 \cdot 8000}{1 \cdot 10^{-5}} \approx 1.6 \cdot 10^{10}$$

This flow is *turbulent* ($Re_x > 4*10^5$).

Using a thermal boundary layer (adapted from notes *9.14* with turbulent boundary layer growth), the surface heat flux, q''_s is described as:

$$q''_s = -k_f \frac{\partial T}{\partial y}\Big|_{y=0} \sim \frac{k_f\left(T_s - T_\infty\right)}{\delta_T}$$

where δ_T is the thermal boundary layer thickness which can be related to the hydrodynamic boundary layer thickness, δ as:

$$\delta_T = \frac{\delta}{\mathrm{Pr}^{1/3}} \sim \delta$$

Turbulent hydrodynamic boundary layers grow as:

$$\delta \sim \frac{x}{\text{Re}_x^{4/5}} \sim x^{1/5}$$

The overall scaling of the surface heat flux to position, x is:

$$q_s'' \sim x^{-1/5}$$

Now relate the heat flux to the local heat transfer coefficient h_x:

$$q_s'' = h_x \left(T_s - T_\infty\right) \sim h_x$$

The relationship between the local heat transfer coefficient and position is:

$$h_x \sim x^{-1/5}$$

Finally using scaling, determine the $h_{x=12000m}$ as:

$$\frac{h_x}{x^{-1/5}} = const \Rightarrow \frac{h_{8000m}}{8000^{-1/5}} = \frac{h_{12000m}}{12000^{-1/5}} \Rightarrow h_{12000m} = B\left(\frac{12000}{8000}\right)^{-1/5} \approx 0.92\ B$$

Using a Nusselt relationship, find the equation for local, constant surface temperature turbulent flow Nusselt number, Nu_x. This is equation (*7.36*) on page 411:

$$Nu_x = 0.0296\ \text{Re}_x^{4/5}\ \text{Pr}^{1/3}$$

where Reynolds number is the only variable to show an x dependence:

$$\text{Re}_x = \frac{Vx}{\nu}$$

Then, from the correlation, relate Nu_x to its x-dependence:

$$Nu_x \sim \text{Re}_x^{4/5} \sim x^{4/5}$$

Recall the definition of Nu_x:

$$Nu_x = \frac{h_x\ x}{k_f} \sim h_x\ x$$

Equating the two previous Nu_x positional dependences gives:

$$x^{4/5} \sim Nu_x \sim h_x\ x \Rightarrow h_x \sim x^{-1/5}$$

Finally using scaling, determine the $h_{x=12000m}$ as:

$$\frac{h_x}{x^{-1/5}} = const \Rightarrow \frac{h_{8000m}}{8000^{-1/5}} = \frac{h_{12000m}}{12000^{-1/5}} \Rightarrow h_{12000m} = B\left(\frac{12000}{8000}\right)^{-1/5} \approx 0.92\ B$$