University of California, Berkeley

Math 1A, Section 3 (Prof. Simić), Fall 2011

Midterm 2 Solutions

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	Score
1	20
2	20
3	20
4	20
5	20
Total	100

Explain your work

1. (20 points) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function and define

$$h(x) = f(x^2) - f\left(\frac{1}{x^2}\right) + f(e^{2(x-1)}),$$

for $x \neq 0$. If $f'(1) = 1$, compute $h'(1)$.

Solution: Using the Chain Rule, we obtain:

$$h'(x) = 2xf'(x^2) + \frac{2}{x^3}f'\left(\frac{1}{x^2}\right) + 2e^{2(x-1)}f'(e^{2(x-1)}).$$

Therefore,

$$h'(1) = 2f'(1) + 2f'(1) + 2f'(1) = 6.$$

2. (20 points) A curve C is defined by the equation

$$x^4 + y^4 = \cos^4 y + xy.$$

Find the equation of the tangent line to C at the point of intersection of C with the positive x-axis.

Solution: When $y = 0$, the equation becomes

$$x^4 = 1.$$

The only positive solution is 1, so the intersection of C with the positive x-axis is the point $(1, 0)$.

Differentiating implicitly and using the Chain Rule, we obtain

$$4x^3 + 4y^3y' = -4\cos^3 y(\sin y)y' + y + xy'.$$

Solving for y' we obtain

$$y' = \frac{y - 4x^3}{4y^3 + 4\sin y\cos^3 y - x}.$$

At the point $(1, 0)$, we have $y'(1) = 4$. Therefore, the equation of the tangent line there is

$$\boxed{y = 4(x - 1).}$$

3. (20 points) (a) Show that the equation $x^3 + 3x + 2 = 0$ has a unique root and that it lies in the interval $(-1, 0)$.

(b) Find the absolute extrema of the function

$$f(x) = \frac{x^3 - 1}{x^2 + 1}$$

on the interval $[-1, 2]$.

Solution: (a) Let $g(x) = x^3 + 3x + 2$. Then $g(-1) = -2 < 0$ and $g(0) = 2 > 0$. Since g is continuous, by the Intermediate Value Theorem there exists a number $c \in (-1, 0)$ such that $g(c) = 0$. Since $g'(x) = 3x^2 + 3 > 0$, g is increasing, so $g(x) = 0$ has a unique solution.

(b) Differentiating using the quotient rule, we obtain

$$f'(x) = \frac{x(x^3 + 3x + 2)}{(x^2 + 1)^2}.$$

Therefore, the **critical points** are $x = 0$ and $x = c$, where c is as in part (a). Since $c < 0$, observe that

$$f(c) = \frac{c^3 - 1}{c^2 + 1} < 0.$$

On the other hand, $f' > 0$ on $(-1, c)$ and $f' < 0$ on $(c, 0)$, so $f(c)$ is the maximal value of f on $[-1, 0]$.

Therefore,

$$f(-1) = f(0) = -1 < f(c) < 0 < \frac{7}{5} = f(2).$$

It follows that on the interval $[-1, 2]$, f attains its **absolute maximum** (equal to $7/5$) at 2 and its **absolute minimum** (equal to -1) at -1 and 0.

4. (20 points) (a) If $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function and $f'(x) = c$, for all $x \in \mathbb{R}$, where c is a constant, what can be said about f?

(b) Assume $f''(x) = 0$, for all $x \in \mathbb{R}$. If $f(0) = -1$ and $f'(0) = 1$, compute f.

Solution: (a) We claim that $f(x) = cx + d$, for some constant d. To prove this, set $g(x) = f(x) - cx$. Then

$$g'(x) = f'(x) - c = 0,$$

for all x. By a corollary of the Mean Value Theorem, it follows that $g(x) = d$, for some constant d, and all x. This proves our claim:

$$f(x) = cx + d.$$

(b) Since $f'' = 0$, by the same corollary as above, $f'(x) = c$, for some constant c. But $f'(0) = 1$, so $c = 1$ and thus $f'(x) = 1$, for all x. By part (a), it follows that $f(x) = x + d$, for some constant d. But

$$-1 = f(0) = d,$$

so

$$\boxed{f(x) = x - 1.}$$

5. (20 points) Let

$$f(x) = e^{-x^2+2x}.$$

(a) Find the intervals of monotonicity and extrema of f.

(b) Find the intervals of concavity and inflection points of f.

(c) Find the horizontal asymptotes of f.

(d) Sketch the graph of f.

Solution: (a) Since

$$f'(x) = e^{-x^2+2x}(-2x+2),$$

it follows that the only critical point is $x = 1$, and that $f' > 0$ on $(-\infty, 1)$ and $f' < 0$ on $(1, \infty)$. Thus:

- f is **increasing** on $(-\infty, 1)$;
- f is **decreasing** on $(1, \infty)$;
- f has an **absolute maximum** at $x = 1$ equal to $f(1) = e$.

(b) Differentiating, we obtain

$$f''(x) = 2e^{-x^2+2x}(2x^2 - 4x + 1).$$

The solutions to the equation $2x^2 - 4x + 1 = 0$ are $1 \pm \frac{\sqrt{2}}{2}$, so $2x^2 - 4x + 1 > 0$ (hence $f'' > 0$) on $(-\infty, 1 - \frac{\sqrt{2}}{2})$ and $(1 + \frac{\sqrt{2}}{2}, \infty)$, and $2x^2 - 4x + 1 < 0$ (hence $f'' < 0$) on $(1 - \frac{\sqrt{2}}{2}, 1 + \frac{\sqrt{2}}{2})$. It follows that:

- f is **concave up** on $(-\infty, 1 - \frac{\sqrt{2}}{2})$ and $(1 + \frac{\sqrt{2}}{2}, \infty)$;
- f is **concave down** on $(1 - \frac{\sqrt{2}}{2}, 1 + \frac{\sqrt{2}}{2})$;
- the **inflection points** are at $1 \pm \frac{\sqrt{2}}{2}$.

(c) Since $-x^2 + 2x \to -\infty$, as $x \to \infty$, it follows that $f(x) \to 0$, as $x \to \infty$. Thus $y = 0$ is a **horizontal asymptote** at both $+\infty$ and $-\infty$.

(d) The graph: