University of California, Berkeley

Math 1A, Section 3 (Prof. Simić), Fall 2011

Midterm 1

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	Score
1	20
2	20
3	20
4	20
5	20
Total	100

Explain your work

1. (20 points) Let us write $\exp(x)$ for e^x. Define

$$f(x) = \exp(1 + \exp(2 + 3x)).$$

(a) Find the domain and range of f.

(b) Show that f is 1–1.

(c) Compute f^{-1}.

Solution: (a) Since exp is defined everywhere, the domain of f is all of $\mathbb{R}$.

First note that the range of the function $g(x) = 2 + 3x$ is $\mathbb{R}$. Therefore, the range of $\exp(2 + 3x)$ is $(0, \infty)$. Since $\exp(2 + 3x) > 0$, it follows that $1 + \exp(2 + 3x) > 1$ so $\exp(1 + \exp(2 + 3x)) > e^1 = e$. Therefore, the range of f is (e, ∞).

(b) Suppose that $f(x_1) = f(x_2)$. Since exp is 1–1, we have:

$$\begin{aligned} f(x_1) = f(x_2) &\Rightarrow 1 + \exp(2 + 3x_1) = 1 + \exp(2 + 3x_2) \\ &\Rightarrow \exp(2 + 3x_1) = \exp(2 + 3x_2) \\ &\Rightarrow 2 + 3x_1 = 2 + 3x_2 \\ &\Rightarrow x_1 = x_2. \end{aligned}$$

This shows that f is 1–1.

(c) Let us solve $y = f(x)$ for x, where $y > e$. We have:

$$\begin{aligned} y = f(x) &\Leftrightarrow \ln y = 1 + \exp(2 + 3x) \\ &\Leftrightarrow \ln y - 1 = \exp(2 + 3x) \\ &\Leftrightarrow \ln(\ln y - 1) = 2 + 3x \\ &\Leftrightarrow x = \frac{\ln(\ln y - 1) - 2}{3}. \end{aligned}$$

Therefore,

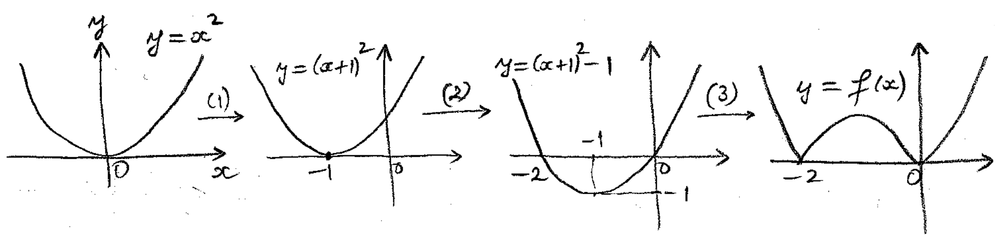
$$\boxed{f^{-1}(y) = \frac{\ln(\ln y - 1) - 2}{3}.}$$

2. (20 points) Sketch the graph of the following functions. Start with the graph of an "easy" function and apply the appropriate transformations.

(a) $f(x) = |x^2 + 2x|$.

(b) $g(x) = \frac{x}{x-2}$.

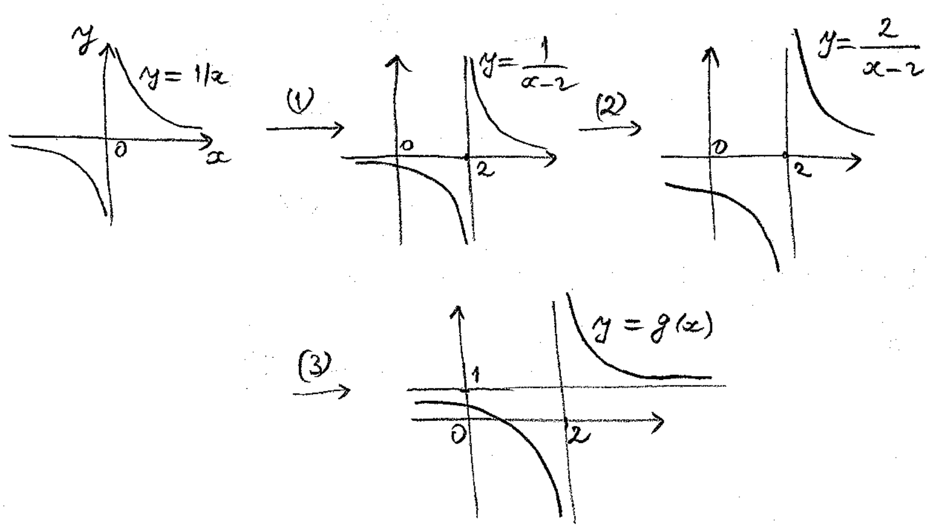
Solution: (a) We write $x^2 + 2x = (x+1)^2 - 1$. Therefore, the graph of f can be obtained from the graph of $y = x^2$ by applying the following transformations: (1) a horizontal shift to the left by 1 unit; (2) a vertical shift down by 1 unit; (3) reflection relative to the x-axis only on the interval $(-2, 0)$, where $x^2 + 2x$ is negative.



(b) Since

$$g(x) = \frac{x - 2 + 2}{x - 2} = 1 + \frac{2}{x-2},$$

the graph of g can be obtained by from the graph of $y = 1/x$ by applying these transformations: (1) a horizontal shift to the right by 2 units; (2) a vertical stretching by a factor of 2; (3) a vertical shift up by 1 unit.



3. (20 points) For each of the following limits, compute it if it exists or explain why it doesn't exist. Use only limit laws and theorems from the book but not informal arguments (such as "this is big and that is big, so their product is big"). Explicitly mention those theorems and limit laws by name (not by their number in the book). In (d), $[x]$ denotes the largest integer $\leq x$.

$$\text{(a)} \quad \lim_{x\to\infty} \frac{1+2x+3x^2+4x^3+5x^6}{x^2+x^6}.$$

$$\text{(b)} \quad \lim_{x\to 0} \frac{e^{-1/x^2}}{1+e^{-1/x^2}}$$

$$\text{(c)} \quad \lim_{x\to 2} \frac{x^4+3x^3-10x^2}{x-2}$$

$$\text{(d)} \quad \lim_{x\to 1} x[x].$$

Solution: (a) Dividing the numerator and the denominator by x^6, we obtain:

$$\begin{aligned}
\lim_{x\to\infty} \frac{1+2x+3x^2+4x^3+5x^6}{x^2+x^6} &= \lim_{x\to\infty} \frac{\frac{1}{x^6}+\frac{2}{x^5}+\frac{3}{x^4}+\frac{4}{x^3}+5}{\frac{1}{x^4}+1} \\
&= \frac{\lim_{x\to\infty}\left(\frac{1}{x^6}+\frac{2}{x^5}+\frac{3}{x^4}+\frac{4}{x^3}+5\right)}{\lim_{x\to\infty}\left(\frac{1}{x^4}+1\right)} \qquad (1)\\
&= 5. \qquad (2)
\end{aligned}$$

In (1) we used the quotient law and in (2) we used the sum law and the fact that $c/x^n \to 0$, as $x \to \infty$, for any constant c and any $n > 0$.

(b) As $x \to 0$, $-1/x^2 \to -\infty$, so $e^{-1/x^2} \to 0$. Therefore,

$$\begin{aligned}
\lim_{x\to 0} \frac{e^{-1/x^2}}{1+e^{-1/x^2}} &= \frac{\lim_{x\to 0} e^{-1/x^2}}{\lim_{x\to 0}(1+e^{-1/x^2})} \qquad (3)\\
&= \frac{0}{1+0} \qquad (4)\\
&= 0.
\end{aligned}$$

In (3) we used the quotient law and in (4) we used the sum law.

(c) Since

$$x^4+3x^3-10x^2 = x^2(x+5)(x-2),$$

it follows that

$$\begin{aligned}\lim_{x\to 2}\frac{x^4+3x^3-10x^2}{x-2} &= \lim_{x\to 2}\frac{x^2(x+5)(x-2)}{x-2}\\ &= \lim_{x\to 2} x^2(x+5)\\ &= \lim_{x\to 2} x^2\cdot\lim_{x\to 2}(x+5) \qquad (5)\\ &= 28.\end{aligned}$$

In (5) we used the product law.

(d) The limit **does not exist**. Indeed:

$$\lim_{x\to 1-} x[x] = \lim_{x\to 1-} x\cdot 0 = 0,$$

whereas

$$\lim_{x\to 1+} x[x] = \lim_{x\to 1-} x\cdot 1 = 1.$$

Since the one-sided limits are different, the (full) limit doesn't exist.

4. (20 points) (a) Using limit laws and theorems from the book, compute

$$\lim_{x \to 0} x^2 \cos \frac{1}{x}.$$

(b) Using the precise (ε-δ) definition of the limit, prove that your answer in (a) is correct. (**Hint:** cosine is bounded.)

Solution: (a) Since

$$-1 \leq \cos \frac{1}{x} \leq 1,$$

for all $x \neq 0$, it follows that

$$-x^2 \leq x^2 \cos \frac{1}{x} \leq x^2.$$

Since both $-x^2$ and x^2 approach 0, as $x \to 0$, by the Squeeze Theorem we have

$$\lim_{x \to 0} x^2 \cos \frac{1}{x} = 0. \tag{6}$$

(b) Let $\varepsilon > 0$ be arbitrary. Define $\delta = \sqrt{\varepsilon}$ and assume $0 < |x| < \delta$. Again using the fact that $|\cos| \leq 1$, we obtain

$$\begin{aligned} |x^2 \cos \frac{1}{x}| &\leq x^2 \\ &< \delta^2 \\ &= \varepsilon. \end{aligned}$$

This proves (6).

5. (20 points) Let

$$f(x) = \begin{cases} \arctan|x| & \text{if} \quad x < 0, \\ \arcsin(x-1) & \text{if} \quad 0 \le x < 2, \\ \frac{\pi}{4}x & \text{if} \quad x \ge 2. \end{cases}$$

(a) Find all the points where f is discontinuous.

(b) Explain why f is continuous at all other points.

(c) Approximately sketch the graph of f.

Solution: (a) and (b): On the intervals $(-\infty, 0)$, $(0, 2)$ and $(2, \infty)$ f is an **elementary function**, therefore continuous. Let us check if f is continuous at 0 and 2.

We have:

$$\lim_{x \to 0-} f(x) = \lim_{x \to 0-} \arctan(-x) = \arctan 0 = 0,$$

whereas

$$\lim_{x \to 0+} f(x) = \lim_{x \to 0+} \arcsin(x-1) = \arcsin(-1) = -\frac{\pi}{2}.$$

Since the one-sided limits at 0 are different, f does not have a limit there. Thus f is discontinuous at 0.

Next,

$$\lim_{x \to 2-} f(x) = \lim_{x \to 2-} \arcsin(x-1) = \arcsin 1 = \frac{\pi}{2},$$

and

$$\lim_{x \to 2+} f(x) = \lim_{x \to 2+} \frac{\pi}{4}x = \frac{\pi}{2}.$$

Thus $f(x) \to \pi/2$, as $x \to 2$. Since $f(2) = \frac{\pi}{4}2 = \pi/2$, it follows that f is continuous at 2.

(c) The graph of f looks approximately like this: