Solutions to the First Midterm Exam – Multivariable Calculus

Math 53, February 25, 2011. Instructor: E. Frenkel

1. Consider the curve in $\mathbb{R}^2$ defined by the equation

$$r = \cos(2\theta).$$

(a) Sketch this curve.

(b) Find the area of the region enclosed by one loop of this curve.

$$\frac{1}{2}\int_{-\pi/4}^{\pi/4} \cos^2(2\theta)d\theta = \frac{1}{4}\int_{-\pi/4}^{\pi/4}(1+\cos(4\theta))d\theta = \frac{\pi}{8}.$$

2. (a) Find an equation of the surface consisting of all points in $\mathbb{R}^3$ that are equidistant from the point $(0,0,1)$ and the plane $z = 2$.

The distance from a point $P = (x, y, z)$ to the point $(0,0,1)$ is $\sqrt{x^2 + y^2 + (z-1)^2}$, and the distance to the plane $z = 2$ is $z - 2$. Hence we obtain the equation

$$\sqrt{x^2 + y^2 + (z-1)^2} = z - 2,$$

which gives

$$x^2 + y^2 + (z-1)^2 = (z-2)^2,$$

and hence

$$z = -\frac{x^2}{2} - \frac{y^2}{2} + \frac{3}{2}.$$

(b) Sketch this surface. What is it called?

This is an elliptic paraboloid which goes downward along the z axis.

3. Show that the function $\dfrac{x^{50}y^{50}}{x^{100} + y^{200}}$ does not have a limit at $(x, y) = (0, 0)$.

First let's approach $(0,0)$ along the y-axis. Then $x = 0$ and $y \neq 0$, so we have along this path $0/y^{200} = 0$ which has limit 0.

Now let's approach $(0,0)$ along the line $x = y$. Then we obtain $x^{100}/(x^{100} + x^{200}) = 1/(1 + x^{100})$ which has the limit 1 as $x \to 0$.

Since the function has two diferent limits along two different lines approaching $(0,0)$, the limit of this function at $(0,0)$ does not exist.

4. Consider the function $f(x, y) = x\cos(y) + y^2e^x + x$.

(a) Find the differential of this function.

$df = (\cos(y) + y^2e^x + 1)dx + (-x\sin(y) + 2ye^x)dy.$

(b) Find an equation of the tangent plane to the graph of this function at the point $(0, \pi, \pi^2)$.

Substituting $x = 0, y = \pi, dx = x - 0, dy = (y - \pi), df = z - \pi^2$, we obtain the equation

$$z - \pi^2 = \pi^2 x + 2\pi(y - \pi).$$

5. Suppose we need to know an equation of the tangent plane to a surface S at the point $P = (1, 3, 2)$. We don't have an equation for S, but we know that the curves

$$\mathbf{r}_1(t) = \langle 1 + 5t, 3 - t^2, 2 + t - t^3 \rangle,$$
$$\mathbf{r}_2(s) = \langle 3s - 2s^2, s + s^3 + s^4, s - s^2 + 2s^3 \rangle$$

both lie in S. Find an equation of the tangent plane to S at the point P.

The point P corresponds to $t = 0, s = 1$.

We find tangent vectors to the two curves:

$$\mathbf{v}_1 = \mathbf{r}_1'(0) = \langle 5, 0, 1 \rangle,$$
$$\mathbf{v}_2 = \mathbf{r}_2'(1) = \langle -1, 8, 5 \rangle.$$

Their cross product is the noral vector to the plane containing both of them:

$$\langle 5, 0, 1 \rangle \times \langle -1, 8, 5 \rangle = \langle -8, -26, 40 \rangle.$$

Hence the following is an equation of the tangent plane:

$$-8(x - 1) - 26(y - 3) + 40(z - 2) = 0.$$