Physics 137B: Midterm 1 Solutions September 27, 2011

Problem 1: Variational Method (5 points)

Use a Gaussian trial function to obtain the lowest upper bound you can on the ground state energy of the linear potential $V(x) = \alpha|x|$.

Hint: The following integrals may be useful:

$$\begin{aligned} \int_0^\infty dx e^{-yx^2} x^2 &= \frac{\sqrt{\pi}}{4} y^{-\frac{3}{2}} \\ \int_0^\infty dx e^{-yx^2} &= \frac{\sqrt{\pi}}{2\sqrt{y}} \end{aligned} \tag{1}$$

Solution

$$\langle V \rangle = 2\alpha A^2 \int_0^\infty x e^{-2bx^2} dx = 2\alpha A^2 \left(-\frac{1}{4b} e^{-2bx^2} \right)_0^\infty = \frac{\alpha A^2}{2b} \tag{2}$$

The normalization can be computed

$$1 = A^2 \int_{-\infty}^\infty e^{-2bx^2} dx = A^2 \sqrt{\frac{\pi}{2b}} \tag{3}$$

Therefore, $A^2 = \sqrt{\frac{2b}{\pi}}$, and so $\langle V \rangle = \frac{\alpha}{\sqrt{2b\pi}}$.
The expectation value of the double derivative

$$\langle \frac{\partial^2}{\partial x^2} \rangle = A^2 \int_{-\infty}^\infty e^{-bx^2} (4b^2x^2 - 2b) e^{-bx^2} dx = A^2 4b^2 2 \frac{\sqrt{\pi}}{4} (2b)^{-3/2} - 2b = \sqrt{\frac{2b}{\pi}} 2b^2 \sqrt{\pi} (2b)^{-3/2} - 2b = -b \tag{4}$$

Therefore, the Hamiltonian expectation value reads

$$\langle H \rangle = \frac{\hbar^2 b}{2m} + \frac{\alpha}{\sqrt{2b\pi}}. \tag{5}$$

Minimizing with respect to b, we obtain

$$\frac{\partial H}{\partial b} = \frac{\hbar^2}{2m} - \frac{1}{2} \frac{\alpha}{\sqrt{2\pi}} b^{-3/2} \to b^{3/2} = \frac{\alpha}{\sqrt{2\pi}} \frac{m}{\hbar^2};\ b = \left(\frac{\alpha m}{\sqrt{2\pi}\hbar^2} \right)^{2/3} \tag{6}$$

$$H_{min} = \frac{\hbar^2}{2m} \left(\frac{\alpha m}{\sqrt{2\pi}\hbar^2} \right)^{2/3} + \frac{\alpha}{\sqrt{2\pi} \left(\frac{\alpha m}{\sqrt{2\pi}\hbar^2} \right)^{-1/3}} = \frac{3}{2} \left(\frac{\alpha^2 \hbar^2}{2\pi m} \right)^{1/3} \tag{7}$$

Problem 2: WKB Approximation (5 points)

For spherically symmetrical potentials we can apply the WKB approximation to the radial part. In the case $l = 0$ the answer is

$$\int_0^{r_0} p(r) dr = (n - 1/4)\pi\hbar, \tag{8}$$

where r_0 is the turning point (in effect, we treat $r = 0$ as an infinite wall). Exploit this formula to estimate the allowed energies of a particle in the logarithmic potential

$$V(r) = V_0 \ln(r/a) \tag{9}$$

(for constants V_0 and a). Treat only the case $l = 0$. Show that the spacing between the levels is independent of mass.

Hint: The following integral may be useful:

$$\int_0^{r_0} \sqrt{\log \frac{r_0}{r}} dr = r_0 \frac{\sqrt{\pi}}{2} \tag{10}$$

<u>Solution</u>
$E = V_0 \ln \frac{r_0}{a}$ defines r_0. The quantization condition reads

$$(n - 1/4)\pi\hbar = \int_0^{r_0} \sqrt{2m(V_0 \ln \frac{r_0}{a} - V_0 \ln r/a)} dr = \sqrt{2mV_0} \int_0^{r_0} \sqrt{\ln(r_0/r)} = \sqrt{2mV_0} r_0 \frac{\sqrt{\pi}}{2} \tag{11}$$

Solving for r_0, we obtain

$$r_0 = (n - 1/4)\hbar \sqrt{\frac{2\pi}{mV_0}} \tag{12}$$

Therefore, the energy levels are

$$E_n = V_0 \ln[\frac{(n - 1/4)\hbar}{a} \sqrt{\frac{2\pi}{mV_0}}] = V_0 \ln(n - 1/4) + V_0 \ln[\frac{\hbar}{a} \sqrt{\frac{2\pi}{mV_0}}] \tag{13}$$

The spacing is given by

$$E_m - E_n = V_0 \ln \frac{m - 1/4}{n - 1/4} \tag{14}$$

which is independent of mass.

<u>Problem 3: Rotating Rigid Body (7 points)</u>
Consider an electrically neutral body that can rotate about a fixed axis, with moment of inertia I. The Hamiltonian for the system is

$$H_0 = \frac{L^2}{2I}, \tag{15}$$

where L is the angular momentum.

(a) What are the possible energy values and what is the degeneracy for each energy eigenvalue?

(b) Now suppose that an electron is embedded in the rotating body at some fixed position off the axis, and an equal positive charge is placed on the axis. Correspondingly, we add a new term to our Hamiltonian, of the form
$H' = \lambda S \cdot L$
where λ is small and S is the electron spin operator. Use first order perturbation theory to determine the energies and degeneracies of the $l = 0$ and $l = 1$ eigenstates.

Solution

a) We must have the eigenstates of L^2 with eigenvalue $\hbar^2 l(l+1)$, irrespective of m_L. Therefore, the eigenvalues are

$$E_{lm} = \frac{\hbar^2 l(l+1)}{2I} \tag{16}$$

The degeneracies are $2l+1$ for each. b) As in the case of spin-orbit coupling, the Hamiltonian perturbation is in the basis of J^2, L^2, S^2. To see this, we write

$$\vec{L}\cdot\vec{S} = \frac{1}{2}(J^2 - L^2 - S^2) \tag{17}$$

Now we can find the perturbed energies to first order:

for l = 0

j must be $1/2$ so

$$\langle l=0, m=0|H'|l=0, m=0\rangle = \frac{1}{2}(\hbar^2\frac{1}{2}\frac{3}{2} - 0 - \hbar^2\frac{1}{2}\frac{3}{2}) = 0 \tag{18}$$

so there is no change in the energy of the ground state. The degeneracy is
$d = \underbrace{(2l+1)}_{spatial}\underbrace{2}_{spin} = 2.$

for l = 1

j can be $1/2, 3/2$. For $j = 3/2$:

$$\langle j=3/2, m_j|H'|j=3/2, m_j\rangle = \frac{1}{2}(\hbar^2\frac{3}{2}\frac{5}{2} - \hbar^2(1)(2) - \hbar^2\frac{1}{2}\frac{3}{2}) = \frac{1}{2}\hbar^2 \tag{19}$$

so the new energy is

$$E = \frac{2\hbar^2}{2I} + \frac{1}{2}\hbar^2\lambda = \frac{\hbar^2}{I} + \frac{1}{2}\hbar^2\lambda \tag{20}$$

The degeneracy is counted by counting the possible m_j's, that's 4 states.
For $j = 1/2$:

$$\langle j=1/2, m_j|H'|j=1/2, m_j\rangle = \frac{1}{2}(\hbar^2\frac{1}{2}\frac{3}{2} - \hbar^2(1)(2) - \hbar^2\frac{1}{2}\frac{3}{2}) = -\hbar^2 \tag{21}$$

so the new energy is

$$E = \frac{\hbar^2}{I} - \hbar^2\lambda \tag{22}$$

The degeneracy is counted by counting the possible m_j's, that's 2 states. Overall, the degeneracy is 6 for the $l = 1$ state, as expected from the simple formula above.

Problem 4: Interaction with a magnetic field (8 points)

Consider a spin-1/2 particle with gyromagnetic ratio γ in a magnetic field $\vec{B} = B'\hat{x} + B_0\hat{z}$, where $\hat{x}$ and $\hat{z}$ are mutually orthogonal vectors of unit length and $B_0 \gg B'$. Treating B' as a perturbation, $H' = -\gamma S_x B'$ and $H_0 = -\gamma S_z B_{0,z}$, calculate the first- and second-order shifts in energy and first-order shift in wave function for the ground state.

Solution

The first-order wave function correction and first- and second-order energy correction are given by, respectively,

$$\psi_n^{(1)} = \sum_{m\neq n} \frac{\psi_m^{(0)}\langle\psi_m^{(0)}|H'|\psi_n^{(0)}\rangle}{E_n^0 - E_m^0}, \tag{23}$$

$$E_n^{(1)} = \langle\psi_n^{(0)}|H'|\psi_n^{(0)}\rangle \tag{24}$$

$$E_n^{(2)} = \sum_{m\neq n} \frac{|\langle\psi_m^{(0)}|H'|\psi_n^{(0)}\rangle|^2}{E_n^0 - E_m^0} \tag{25}$$

To evaluate these, we first have to find the zeroth-order energy levels and eigenfunctions. They are given as eigenstates of the spin operator in the z-direction with spin quantum number $m_s = \pm 1/2$, respectively. Correspondingly, the energies are

$$E_\pm = \mp\frac{1}{2}\hbar\gamma B_0 \tag{26}$$

for the eigenstates $\psi_\pm$ with spin eigenvalue $m_s = \pm 1/2$, respectively. Next we have to evaluate the matrix element of the perturbation. The spin operator in the x-direction is given as $S_x = \frac{\hbar}{2}\sigma_x = \frac{\hbar}{2}\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Therefore, the matrix elements are given as

$$\begin{aligned}
\langle\psi_+|H'|\psi_+\rangle &= 0 \\
\langle\psi_-|H'|\psi_-\rangle &= 0 \\
\langle\psi_+|H'|\psi_-\rangle &= \frac{-\gamma\hbar B'}{2} \\
\langle\psi_-|H'|\psi_+\rangle &= \frac{-\gamma\hbar B'}{2}
\end{aligned} \tag{27}$$

It follows that the first-order shift to the energy of the states is zero, because diagonal perturbations vanish. The first-order correction to the eigenstate and second-order correction to the energy is given as

$$\psi_+^{(1)} = \frac{\psi_-^{(0)}\langle\psi_-^{(0)}|H'|\psi_+^{(0)}\rangle}{E_+^0 - E_-^0} = \psi_-^{(0)}\frac{\frac{-1}{2}\gamma\hbar B'}{-\hbar\gamma B_0} = \psi_-^{(0)}\frac{B'}{2B_0} \tag{28}$$

$$\psi_-^{(1)} = \frac{\psi_+^{(0)}\langle\psi_+^{(0)}|H'|\psi_-^{(0)}\rangle}{E_-^0 - E_+^0} = \psi_+^{(0)}\frac{\frac{-1}{2}\gamma\hbar B'}{\hbar\gamma B_0} = -\psi_+^{(0)}\frac{B'}{2B_0} \tag{29}$$

$$E_+^{(2)} = \frac{|\langle\psi_-^{(0)}|H'|\psi_+^{(0)}\rangle|^2}{E_+^0 - E_-^0} = \frac{(\frac{1}{2}\gamma\hbar B')^2}{-\hbar\gamma B_0} = \frac{-\hbar\gamma(B')^2}{4B_0} \tag{30}$$

$$E_-^{(2)} = \frac{|\langle\psi_+^{(0)}|H'|\psi_-^{(0)}\rangle|^2}{E_-^0 - E_+^0} = \frac{(\frac{1}{2}\gamma\hbar B')^2}{\hbar\gamma B_0} = \frac{\hbar\gamma(B')^2}{4B_0} \tag{31}$$