STATISTICS 134 | S.M. STOYANOV

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MIDTERM 2

NAME: ____________________ SID: ____________________

Please show ALL WORK AND REASONING for ALL the problems. Unless indicated otherwise, please work the problem through to a numerical answer. You may use a calculator and a handwritten page of notes.

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1. Let μ denote the true average caffeine content in each 12 oz. can of a certain cola. A sample of size n cans of this cola is collected. The amount of caffeine in each is a random variable with mean μ and standard deviation 0.1 mg. How large should n be in order to ensure that the probability that sample average caffeine content is within 0.02 mg of the true average caffeine content is at least 0.95. [/3]

$\boxed{\text{Using Chebyshev}}$ $\sigma^2 = (0.1)^2/n = \frac{0.01}{n}$

$$P(|\bar{X}-\mu| \le 0.02) \ge 1 - \frac{0.01}{n(0.02)^2} \ge 0.95$$

$$\Rightarrow n \ge \frac{0.01}{(0.05)(0.0004)} = 500$$

n should be at least 500, if we don't assume anything.

$\boxed{\text{Using CLT}}$ $\therefore$ $\bar{X} \sim N(\mu, (0.1)^2/n)$

$$P(|\bar{X}-\mu| \le 0.02) \ge 0.95$$

$$\Rightarrow P(-0.02 \le \bar{X}-\mu \le 0.02) \ge 0.95$$

$$\Rightarrow P\left(-\frac{0.02\sqrt{n}}{0.1} \le Z \le \frac{0.02\sqrt{n}}{0.1}\right) \ge 0.95$$

$$2\Phi\left(\frac{0.02\sqrt{n}}{0.1}\right) - 1 \ge 0.95$$

$n \ge 96.04$, so n should be at least 97

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2. Let X be normally distributed with mean 80 and standard deviation 2. Find a constant c such that $P(|X-80| > c) \approx 1/3$. [/3]

$$X \sim N(80, 4)$$

Note: We know c will be about 2 since about 2/3 of the area under a normal curve is within 1 SD.

$$P(|X-80| > c) \approx 1/3 \Rightarrow P(|X-80| < c) \simeq \frac{2}{3}$$

$$\Rightarrow 2\Phi\left(\frac{c}{2}\right) - 1 \simeq \frac{2}{3}$$

$$\Phi(c/2) = 5/6$$

$$\boxed{c = 1.93}$$

3. Let X be a discrete random variable with moment generating function $M_X(t) = .3 + .2e^{2.5t} + .5e^{3t}$. Find the probability distribution of X, and $Var(X)$. [/4]

Values taken by X

$$M_X(t) = E(e^{tX}) = 0.3 + 0.2e^{2.5t} + 0.5e^{3t}$$

probabilities

$$X = \begin{cases} 0 & \text{w.p. } 0.3 \\ 2.5 & \text{w.p. } 0.2 \\ 3 & \text{w.p. } 0.5 \end{cases}$$

$$E(X) = 0 + (2.5)(.2) + (3)(0.5) = 2$$

$$Var(X) = E(X^2) - (E(X))^2 = [(0.3)(0) + (1.25)(0.2) + (9)(0.5) - 4] = \boxed{1.75}$$

$$Var(X) = M_X''(0) - (M_X'(0))^2 = 5.75 - 4 = \boxed{1.75}$$

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4. Let Z be a standard normal random variable (with mean 0 and variance 1). Let X be Bernoulli, taking values 1 and -1 with equal probability, and let $Y = XZ$. Is Y standard normal as well? Explain your answer by either showing that it must be, or that it cannot be. [/4]

$$Z \sim N(0,1), \quad X \simeq \begin{cases} -1 & \text{w.p. } \frac{1}{2} \\ 1 & \text{w.p. } \frac{1}{2} \end{cases} \Rightarrow Y = \begin{cases} Z & \text{w.p. } \frac{1}{2} \\ -Z & \text{w.p. } \frac{1}{2} \end{cases}$$

$(Y = XZ)$

If we can show that the moment generating function of 2 random variables are the same, (when the mgf exists and is finite in a region containg 0), then the distributions coincide!!

$$E(e^{tY}) = \sum_y e^{ty} P(Y=y)$$

$$= \frac{1}{2} E(e^{tZ}) + \frac{1}{2} E(e^{-tZ})$$

$$= \frac{1}{2} M_Z(t) + \frac{1}{2} M_Z(-t)$$

$$\frac{1}{2} e^{t^2/2} + \frac{1}{2} e^{t^2/2} = e^{\frac{t^2}{2}}$$

So Y is std normal

Alternatively

$$P(Y \le y) = P(XZ \le y)$$

$$= P(XZ \le y \mid X = 1) P(X=1) + P(XZ \le y \mid X = -1) P(X = -1)$$

$$= P(Z \le y)\frac{1}{2} + P(-Z \le y)\frac{1}{2}$$

$$= P(Z \le y)\frac{1}{2} + P(Z \ge -y)\frac{1}{2}$$

By symmetry, these are the same.

5. If X and Y are jointly distributed discrete random variables with $P(X = i, Y = j) = \frac{ij}{18}$ for $i = 1, 2$ and $j = 1, 2, 3$, find

(a) $P(X + Y > 3)$.

[/3]

$$P(X+Y>3) = P(X+Y=4) + P(X+Y=5)$$

$$= P(X=1, Y=3) + P(X=2, Y=2) + P(X=2, Y=3)$$

$$= \frac{3}{18} + \frac{4}{18} + \frac{6}{18} = \frac{13}{18}$$

(b) The marginal distribution of Y.

[/3]

$$f_Y(y) = \sum_x f(x,y) = \sum_{x=1}^{2} \frac{xy}{18} = \frac{y}{18}(1+2)$$

$$= \frac{3y}{18} = \frac{y}{6}$$

$$f_Y(y) = \frac{y}{6}, \quad y = 1, 2, 3.$$

6. Let the discrete random variables U_1, U_2 be independent, and uniform on $\{1, 2, \ldots, n\}$, (so $P(U_i = k) = 1/n$), and let $M = \max(U_1, U_2)$. Find the distribution of M. [/3]

$$P(M=k) = P(M \leq k) - P(M < k) = P(M \leq k) - P(M \leq k-1)$$

$$P(M \leq k) = P(U_1 \leq k \ \& \ U_2 \leq k)$$
$$= P(U_1 \leq k) P(U_2 \leq k)$$
$$= \frac{k}{n} \cdot \frac{k}{n} = \frac{k^2}{n^2}$$

$$\Rightarrow P(M=k) = \frac{k^2}{n^2} - \frac{(k-1)^2}{n^2} = \boxed{\frac{2k-1}{n^2}}, \quad k=1,2,\ldots,n$$

Check $\sum_{k=1}^{n} P(M=k) = \sum_{k=1}^{n} \frac{(2k-1)}{n^2} = 1.$ ✓

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7. Suppose that X has Poisson (μ) distribution, and Y has geometric (p) distribution on $\{0, 1, 2, \ldots\}$ independently of X. Find a formula for $P(Y \geq X)$ in terms of μ and p, and evaluate it numerically for $p = 1/2$, $\mu = 1$. [/4]

$$X \sim \text{Poisson}(\mu) \Rightarrow P(X=k) = \frac{e^{-\mu} \mu^k}{k!}$$

$$Y \sim \text{Geom}(p) \Rightarrow P(Y=k) = (1-p)^k p$$

on $\{0,1,2,\ldots\}$

(so Y is like # of failures before 1st success)

$$P(Y \geq x) = \sum_{k=x}^{\infty} P(Y=k) = \sum_{k=x}^{\infty} (1-p)^k p = (1-p)^x$$

$$P(Y \geq X) = \sum_{x=0}^{\infty} P(Y \geq x \mid X=x) P(X=x)$$
$$= \sum_{x=0}^{\infty} P(Y \geq x) P(X=x) \quad \text{(Independence of } X, Y\text{)}$$
$$= \sum_{x=0}^{\infty} (1-p)^x \frac{e^{-\mu} \mu^x}{x!}$$

$$P(Y \geq X) = \sum_{x=0}^{\infty} \frac{e^{-\mu}}{x!} (\mu(1-p))^x = e^{-\mu} \sum_{x=0}^{\infty} \frac{[\mu(1-p)]^x}{x!} = e^{-\mu} e^{\mu(1-p)}$$

For $p = 1/2$, $\mu = 1$ $\quad P(Y \geq X) = e^{-1/2} \approx 0.6065$