STATISTICS 134
FALL 2011

MIDTERM 1

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NAME: Solutions

SID: ____________

Please show ALL WORK AND REASONING for ALL the problems. Unless indicated otherwise, please work the problem through to a numerical answer. You may use a calculator and a handwritten page of notes. The exam is out of 40 points.

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1. How many five-letter code words are possible using the letters in HOUSE if:

   (a) The letters may be repeated? **(2 points)**

   5 choices for each letter of the word, so 5^5 words possible. $\boxed{3125}$

   (b) The letters may not be repeated? **(2 points)**

   $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5! = \boxed{120 \text{ words}}$

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2. A pair of dice are thrown.

   (a) Find the probability that both dice show the same number of spots. **(2 points)**

   P(Both dice show same # of spots)
   $= P(\{(K,K) : 1 \le K \le 6\}) = 6 \cdot \frac{1}{36} = \frac{1}{6}$

   (b) Show that the event that the sum of the spots on the dice is 7 is independent of the number of spots on the first die. **(3 points)**

   Let A_K be the event that first die shows K spots.
   Let B be the event that sum of spots = 7

   $P(A_K B) = P(\text{First die} = K \ \& \text{ sum is seven})$
   $= P((K, 7-K)) = \frac{1}{36}$

   $P(A_K) = \frac{1}{6}, \quad P(B) = \frac{1}{6} \Rightarrow \boxed{P(A_K B) = P(A_K) P(B)}$

3. Show that if A and B are independent events, then A^c and B^c must also be independent. (3 points)

Given: $P(AB) = P(A)P(B)$

$$P(A^cB^c) = P(A^c) - P(A^cB)$$
$$= (1-P(A)) - [P(B) - P(AB)]$$
$$= (1-P(A)) - P(B)(1-P(A))$$
$$= (1-P(A))(1-P(B)) = \underline{P(A^c)P(B^c)}$$

4. A, B and C are mutually independent events that occur with probabilities $P(A) = 0.3$, $P(B) = .2$, $P(C) = 0.5$.

(a) Find the probability that at least one of the events occurs. (2 points)

$$P(\text{at least one}) = 1 - P(\text{none})$$
$$= 1 - P(A^cB^cC^c)$$
$$= 1 - P(A^c)P(B^c)P(C^c) \quad (\text{by } \#3)$$
$$= 1 - (0.7)(0.8)(0.5)$$
$$= \boxed{0.72}$$

(b) Find the probability that exactly 2 of the events occur. (3 points)

$$P(\text{exactly } 2) = P(ABC^c) + P(AB^cC) + P(A^cBC)$$
$$= (0.3)(0.2)(0.5) + (0.3)(0.8)(0.5) + (0.7)(0.2)$$
$$= \boxed{0.22}$$

5. In a game of poker, 5 cards are dealt from a well-shuffled standard deck. (A standard deck has 52 cards: 4 suits, with 13 cards in each suit.)

(a) How many 5-card hands can be dealt? (2 points)

$$\#\text{ of 5 card hands} = \binom{52}{5} = \frac{52!}{5!\,47!} = \frac{52\cdot51\cdot50\cdot49\cdot48}{5\cdot4\cdot3\cdot2\cdot1} = \boxed{2{,}598{,}960}$$

(b) What is the probability that a 5-card hand will contain a full house (3 cards of one value, and 2 of another value)? (2 points)

$$P(\text{Full house}) = \frac{\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}}{\binom{52}{5}} = \frac{13\cdot4\cdot12\cdot6}{2{,}598{,}960}$$
$$= \boxed{0.00144}$$

6. A (biased) coin is flipped until a head appears for the first time. Let X be the number of tails that occur, and let $P(H) = \frac{1}{3}$.

(a) Write down the probability that $X = k$, where $k = 0, 1, 2, \ldots$ (2 points)

If # of tails $= k$, then we have $\underbrace{TTT \ldots T}_{k} H$

$$P(X=k) = \left(\frac{2}{3}\right)^k \left(\frac{1}{3}\right)$$

(b) Find $P(X = 3 | X > 2)$ (3 points)

$$P(X=3 \mid X>2) = \frac{P(\{X=3\} \cap \{X>2\})}{P(X>2)}$$

$$= \frac{P(X=3)}{\underbrace{P(X>2)}_{1-P(X\leq 2)}} = \frac{\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)}{1-\left[\left(\frac{2}{3}\right)^0\left(\frac{1}{3}\right)+\left(\frac{2}{3}\right)^1\left(\frac{1}{3}\right)+\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)\right]} = \frac{\cancel{\frac{8}{27}}\cdot\frac{1}{3}}{\cancel{\frac{8}{27}}}$$

Ans: $\boxed{\frac{1}{3}}$

(c) Now, suppose the coin is flipped until we see **three** heads, so we stop after the **third** head. Let Y be the number of **tails** in this situation.
Write down the probability that $Y = k$, where $k = 0, 1, 2, \ldots$ (3 points)

If $Y=k$, $\underbrace{\_\ \_\ \_\ \_\ \_}_{k \text{ tails},\ 2H}$ H ← Last (third) H.

$$P(Y=k) = \left(\frac{2}{3}\right)^k \left(\frac{1}{3}\right)^3 \cdot \binom{k+2}{2}$$ ← # of ways to arrange 2 H, k tails.

7. Let H denote the part of the population that has tried heroin, and M denote the part of the population that has tried marijuana. Draw a Venn diagram to demonstrate that we can have $P(M|H)$ be close to 1, but $P(H|M)$ be close to 0. (2 points)

$$P(H|M) = \frac{P(HM)}{P(M)} \approx 0$$

$$P(M|H) = \frac{P(HM)}{P(H)} \approx 1$$

Note: $P(HM) \approx P(H)$

8. A man has five coins, two of which are double-headed, one double-tailed, and two that are normal (fair) coins.

(a) He shuts his eyes, picks a coin at random, and tosses it. What is the probability that the **lower face** of the coin is a head? (3 points)

Let HH denote a double-headed coin
TT " " " tailed coin
HT " " normal coin

H_l^1 : Lower face is H on 1st toss

$$P(H_l^1) = P(H_l^1 \mid HH)P(HH) + P(H_l^1 \mid TT)P(TT) + P(H_l^1 \mid HT)P(HT)$$

$$= 1 \cdot \tfrac{2}{5} + 0 + \tfrac{1}{2} \cdot \tfrac{2}{5} = \boxed{3/5}$$

(b) He opens his eyes, and sees that the coin has landed heads. What is the probability that the **lower face** of the coin is a head? (3 points)

H_u^1 : upper face on 1st toss = H.

$$P(H_l^1 \mid H_u^1) = \frac{P(H_l^1 \cap H_u^1)}{P(H_u^1)} = \frac{P(HH)}{P(H_u^1)} = \frac{2/5}{3/5} = \boxed{\frac{2}{3}}$$

From (a)

(c) He tosses the coin again, and sees that it lands heads again. What is the probability that the coin is double-headed? (3 points)

H_u^2 : Lands H on 2nd toss.

We want $P(HH \mid H_u^2 \cap H_u^1) = \dfrac{P(HH \cap \{H_u^2 \cap H_u^1\})}{P(H_u^2 \cap H_u^1)}$

$$= \frac{P(H_u^2 \cap H_u^1 \mid HH)P(HH)}{P(H_u^2 \cap H_u^1 \mid HH)P(HH) + P(H_u^2 \cap H_u^1 \mid HT)P(HT)}$$

$$= \frac{1 \cdot 2/5}{1 \cdot 2/5 + \frac{1}{4} \cdot 2/5} = \frac{1}{1 + \frac{1}{4}} \boxed{\frac{4}{5}}$$

(The TT gives 0 prob)