Mathematics 115 Professor K. A. Ribet

First Midterm Exam September 22, 2011

Please put away all cell phones, books, iPads, iPods, laptops, etc., etc. You may consult a single two-sided sheet of notes. Please write carefully and clearly in complete sentences; take pains to explain what you are doing instead of hoping that the grader will read your mind.

This document contains some quick and dirty "answers" written by Ribet. The intention is not to write up model solutions but rather to give a sense of how things should go.

(8 points) **1.** Calculate the gcd g of 345 and 357. Find integers x and y so that

$$g = 345x + 357y.$$

This is a pretty standard problem that we've done in class and that I hope you know how to do without breaking a sweat. We have

$$357 = 345 + 12, \quad 345 = 28 \cdot 12 + 9, \quad 12 = 9 + 3, \quad 3|9.$$

Thus 3 is the gcd, and moreover

$$12 = 357 - 345, \quad 9 = 345 - 28 \cdot 12 = 29 \cdot 345 - 28 \cdot 357, \quad 3 = 12 - 9 = \cdots = 29 \cdot 357 - 30 \cdot 345.$$

(8 points) **2.** Find the prime factorization of 46!.

Clearly, only the primes ≤ 43 intervene in the factorization. Further, the exponent of p in the prime factorization of 46! is $\lfloor \frac{46}{p} \rfloor$ when $p^2 > 46$, i.e., for $p > 5$. The exponents of 2, 3 and 5 in the factorization are slightly harder to compute; the general formula is $\sum_{i=1}^{\infty} \lfloor \frac{46}{p^i} \rfloor$, as we saw in class. For the (presumably) correct answer, I turned to sage, which gives the factorization

$$46! = 2^{42} \cdot 3^{21} \cdot 5^{10} \cdot 7^{6} \cdot 11^{4} \cdot 13^{3} \cdot 17^{2} \cdot 19^{2} \cdot 23^{2} \cdot 29 \cdot 31 \cdot 37 \cdot 41 \cdot 43.$$

(10 points) **3.** Let p be a prime number with $p \equiv 3 \bmod 4$. Using Wilson's theorem, show that $\left(\frac{p-1}{2}\right)!$ is either $+1$ or -1 mod p.

To me, there are two points: first to show that the square of $\left(\frac{p-1}{2}\right)!$ is 1 and second to point out that the only numbers mod p whose squares are 1 are ± 1. The second issue was stressed in the book and also in lecture, so I hope that you at least allude to the fact that this needs to be proved (and was). For the first issue, see the proof of Theorem 2.12 in the book and notice that “the first factor on the right” in the first displayed equation on page 54 is -1 rather than $+1$ when $p \equiv 3 \bmod 4$.

(10 points) **4.** Suppose that m and n are integers ≥ 2 for which $(\log m)/(\log n)$ is rational, say $\dfrac{\log m}{\log n} = \dfrac{a}{b}$ in lowest terms. Show that there is an integer c so that $m = c^a$ and $n = c^b$.

Cross multiplication yields $b \log m = a \log n$, so that we have $m^b = n^a$ after exponentiation. Let $p_1, \ldots, p_t$ be the prime numbers occurring in either the factorization of m or the factoriation of n. Then we have prime factorizations $m = \prod p_i^{e_i}$, $n = \prod p_i^{f_i}$, where the e_i and f_i are non-negative integers. This gives two prime factorizations of $m^b = n^a$, namely $\prod p_i^{be_i}$ and $\prod p_i^{af_i}$. The two must be the same by the fundamental theorem of arithmetic. In concrete terms, we have $be_i = af_i$ for each i.

For each i, we note that b divides af_i and that $\gcd(a, b) = 1$ (because the fraction was in lowest terms). Hence b divides f_i; say $f_i = bs_i$ for some s_i. Similarly, one has $e_i = ar_i$ for some r_i. Then $abr_i = be_i = af_i = abs_i$; this shows that $r_i = s_i$. We put $c = \prod p_i^{r_i}$ and see that $c^a = m$ and $c^b = n$ as required.

5. When $f(x)$ is a function of a real variable, we have written $\Delta f(x) = f(x+1) - f(x)$ and have defined $\Delta^k f(x)$ by the recursive formula $\Delta^k f = \Delta(\Delta^{k-1} f)$.

(3 points) **a.** Show that $\Delta\binom{x}{i} = \binom{x}{i-1}$ for $i \geq 1$.

The polynomial identity to be proved is

$$\frac{(x+1)x(x-1)\cdots(x-i+2)}{i!} - \frac{x(x-1)(x-2)\cdots(x-i+1)}{i!} = \frac{x(x-1)\cdots(x-i+2)}{(i-1)!}.$$

All terms in the fraction on the right-hand side appear in both fractions on the left-hand side. After we factor out those terms from the fractions on the left-hand side, what remains in the first fraction on the left-hand side is $\frac{x+1}{i}$; for the second fraction, what remains is $\frac{x-i+1}{i}$. The difference of these two terms is $\frac{i}{i} = 1$.

(6 points) **b.** If $f(x) = \sum_{i=0}^{n} c_i \binom{x}{i}$, show that $c_i = \Delta^i f(x)\,|_{x=0}$ for $i = 0, \ldots, n$.

One point is that $\binom{x}{i}$ vanishes at $x = 0$ for $i > 0$. Setting $x = 0$ in the given equation $f(x) = \sum_{i=0}^{n} c_i \binom{x}{i}$, we thus get $f(0) = c_0$. Because $\Delta c = 0$ when c is a constant, part (a) yields $\Delta f(x) = \sum_{i=1}^{n} c_i \binom{x}{i-1}$. Setting $x = 0$ again, we obtain $\Delta f(0) = c_1$. By repeated “differentiation” (i.e., application of Δ), we obtain analogous formulas for c_2, c_3, etc.