Solutions

October 17, 2011, ME106, Mid-term 2 **Check your dimensions!**

The figure shows a siphon with a uniform cross-sectional area A. The bottom of both tanks are at $z = 0$; the water surfaces in the tank on the right and left are respectively $z = h$ and $z = H$. The bottoms of the siphon on the right and left are respectively at $z = L$ and $z = B$. The flow is steady in time. Water with constant density ρ passes through the siphon from left to right. The bottom of the right side of the siphon is high above the surface of the water of the tank on the right side. Just below the siphon at point **6** at $z = L$, the water in the exiting stream has atmospheric pressure P_{atm}. *All pressures in the exam are absolute pressures.* The water exiting from the bottom of the tank on the right goes through a pipe and a pump and is returned into the tank on the left such that the surface of the water level there is at $z = H$. The tank on the left is sufficiently big that the velocity of the water at point **1** at $z = H$ is approximately zero. The rate (mass of water per unit time) that the pump transports from right to left in the pipe is $\dot{M}$, which is the same rate that water mass is transported through the siphon. The pressures at point **2** (just beneath the siphon on the left) and at point **3** (just inside the siphon on the left) are the same. The pressures and velocities at point **6** (just beneath the siphon on the right) and at point **5** (just inside the siphon the right) are the same.

1) Assuming that the siphon is completely filled with water, and given the values of ρ, $\dot{M}$, A, L, B, P_{atm}, and gravity g, find the magnitude of the velocity V_6 of the water that exits the siphon. (Hint: Use the fact that the rate at which mass exits the siphon is $\dot{M}$.) Now either find the value of H such that the flow is steady, or show why it is impossible to determine the value of H from the given information. (It is possible that I have given you *more* information than you need, and that some information is extraneous.)

Solution: The rate at which mass exits the siphon at point **6** is the mass flux ρV_6 multiplied by the cross-sectional area A of the siphon that is normal to the streamlines, so $\dot{M} = \rho A V_6$, or equivalently $V_6 = \dot{M}/(\rho A)$. Now use Bernoulli between points **1** and **6**. The Bernoulli function for a constant-density, steady flow is $V^2/2 + gz + P/\rho$. At **1**, $P = P_{atm}$, $gz = gH$, and $V = 0$. Note that H is unknown. Thus, the Bernoulli function of the water at **1** is $(gH + P_{atm}/\rho)$. At **6**, $P = P_{atm}$, $gz = gL$, and $V_6 = \dot{M}/(\rho A)$. Thus, the Bernoulli function of the water at **6** is $[(V_6)^2/2 + gL + P_{atm}/\rho]$. Equating the two Bernoulli functions, we obtain $H = L + (V_6)^2/(2g) = L + (\dot{M})^2/(2g\rho^2 A^2)$. The expression on the right side of this equation has dimensions of "length", as required.

Now assume that for $z \leq T$, the siphon is filled with *liquid* water, and for $z > T$ it is filled with water vapor. (This situation of a liquid/gas interface is analogous to the manometer or U-tube apparatus we discussed in class for measuring the atmospheric pressure where in the closed section of the U-tube above the liquid surface there was vapor.) The vapor has pressure P_{vapor} within the siphon along the liquid/gas interface, including point **4**. The mass transported in the siphon and pipe are both $\dot{M}$, where we assume that the transport is all due to *liquid* water rather than the water *vapor*. The cross-sectional area (in the direction normal to the streamlines) of the liquid water in the siphon at point **4** is a (with $0 < a < A$). At points **3** and **5** the area is A. Thus, the water velocities at **3**, **4** and **5** *cannot all be the same.*

2) Find the new answers to question 1, assuming you are give the values of a, P_{vapor}, and T. Do your answers change?

Solution: The rate at which mass exits the siphon at point **6** is the mass flux ρV_6 multiplied by the cross-sectional area A of the siphon that is normal to the streamlines, so $\dot{M} = \rho A V_6$, or equivalently $V_6 = \dot{M}/(\rho A)$. Thus, the value of V_6 does not change from part 1. Again, use Bernoulli between points **1** and **6**. At **1**, $P = P_{atm}$, $gz = gH$, and $V = 0$. Note that H is an unknown and that its value might be different from its value in part 1. Thus, the Bernoulli function of the water at **1** is $(gH + P_{atm}/\rho)$. At **6**, $P = P_{atm}$, $gz = gL$, and $V_6 = \dot{M}/(\rho A)$. Thus, the Bernoulli function of the water at **6** is unchanged from your answer in part 1. Thus, the value of $H = L + (\dot{M})^2/(2g\rho^2 A^2)$ is also unchanged from part 1.

3) Assuming that the siphon is only partially filled with water, and given the values of a, ρ, $\dot{M}$, A, L B, g, P_{atm}, P_{vapor} and H (from part 2), either find the value of T such that the flow is steady, or show why it is impossible to determine the value of T from the given information.

Solution: At point **4**, we use $\dot{M} = \rho a V_4$, or equivalently $V_4 = \dot{M}/(\rho a)$. A streamline connects points **1**, **4** and **6**, so we now exploit that fact by evaluating the Bernoulli function at **4**. At **4**, $P = P_{vapor}$, $gz = gT$, and $V = V_4$, all of which are known. The Bernoulli function at **4** is $[(V_4)^2/2 + gT + P_{vapor}/\rho]$. Equating this value to the value of the Bernoulli function at **1**, we obtain $P_{atm}/\rho + gH = [(V_4)^2/2 + gT + P_{vapor}/\rho]$, or equivalently $T = H + (P_{atm} - P_{vapor})/(g\rho) - (V_4)^2/(2g)$. Substituting our expressions for H and V_4 into the right side of this equation, we obtain $T = [L + (\dot{M})^2/(2g\rho^2 A^2)] + (P_{atm} - P_{vapor})/(g\rho) - (\dot{M})^2/(2g\rho^2 a^2)$, or equivalently $T = L + (\dot{M})^2/(2g\rho^2)(1/A^2 - 1/a^2) + (P_{atm} - P_{vapor})/(g\rho)$.