Date: December 15th, 2007

UNIVERSITY OF CALIFORNIA

College of Engineering

Departments of Mechanical Engineering and Materials Science & Engineering

Fall 2007 Prof. Ritchie

MSE C113/ME C124

Mechanical Behavior of Materials

Final Exam Solutions

Name:____________________________________

SID No.:__________________________________

No.	Total Credit	Credit
1	40	
2	40	
3	40	
4	40	
5	40	
Total	200	

- **Attempt all 5 (five) questions, which are all of equal credit.**
- **Please be neat and concise in your solutions. This makes it easier to follow your train of thought**
- **Stress-intensity solutions and some useful equations are given at the end of the exam sheet**

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Problem 1 (Elasticity and Mechanics)

You are given a piece of ductile polycrystalline metal, such as virgin copper, with uniform cross-sectional area that behaves according to the following constitutive behavior:

$$\sigma_{11} = K\varepsilon_{11}^{n}$$

where σ_{11} and ε_{11} are respectively the uniaxial(true) stress and plastic strain.

a) If this material is pulled in uniaxial tension, draw and label the following on a Stress-Strain graph:
   i) variation in engineering stress vs. engineering strain
   ii) the corresponding variation in true stress vs. true strain
   iii) indicate where necking occurs
   iv) label the ultimate tensile strength, σ_{UTS}

b) Briefly explain (3 sentences or less) the difference between the true stress-strain and the engineering stress-strain diagrams.

c) By stating your assumptions clearly, derive the following relationships:
   i) An expression relating true stress to engineering stress and engineering strain only
   ii) An expression relating true strain to engineering strain only

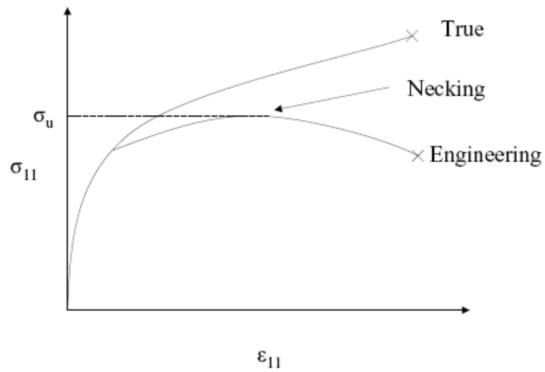
d) Consider an infinite plate loaded so that the stress state in the plate is given by:

$$\sigma_{ij} = \begin{bmatrix} 400 & 0 & 125 \\ 0 & 0 & 0 \\ 125 & 0 & -75 \end{bmatrix}$$

   i) What are the principle stresses and what are the principle axes?
   ii) The yield strength of the material is $\sigma_y = 300$ MPa. Use both the Tresca and Von Mises criteria to determine whether the plate yields.

Solution:

a) Stress-Strain Plot



b) True Stress-Strain compensates for changing cross-sectional area, while Engineering Stress-Strain uses the Original Area.

c) Derive a relationship:

i) An expression relating true stress to engineering stress and engineering strain only

Assume constant volume

$$A_0\ell_0 = A\ell$$

$$\frac{A_0}{A} = \frac{\ell}{\ell_0}$$

$$\varepsilon_{eng} = \frac{\ell - \ell_0}{\ell_0} = \frac{\ell}{\ell_0} - 1$$

$$\frac{\ell}{\ell_0} = \varepsilon_{eng} + 1$$

Also, $\sigma_{eng} = \frac{P}{A_0}$ and $\sigma_T = \frac{P}{A}$

$$\sigma_T = \sigma_{eng}\frac{A_0}{A} = \sigma_{eng}\frac{\ell}{\ell_0}$$

$$\sigma_T = \sigma_{eng}\left(1 + \varepsilon_{eng}\right)$$

ii) an expression relating true strain to engineering strain only.

$$d\varepsilon_T = \frac{d\ell}{\ell}$$

$$\varepsilon_T = \int_{\ell_0}^{\ell}\frac{d\ell}{\ell} = \ln\left(\frac{\ell}{\ell_0}\right)$$

$$\varepsilon_T = \ln\left(1 + \varepsilon_{eng}\right)$$

d) Infinite Plate

i) Use Mohr's circle to get the principle stresses:

σ_{11} = 400 MPa

σ_{33} = -75 MPa

$\sigma_{13} = \sigma_{31}$ = 125 MPa

Center: $C = \frac{\sigma_{11} - \sigma_{33}}{2} = \frac{400 + {}^{-}75}{2} = 162.5\,MPa$

A = σ_{11} – C = 400-162.5 = 237.5 MPa

R = $\sqrt{237.5^2 + 125^2} = 268.4\,MPa$

σ_I = C + R = 162.5 + 268.4 = 430.9 MPa

σ_{III} = C – R = 162.5 – 268.4 = -105.9 MPa

$2\theta \cong \tan^{-1}\left(\frac{125}{237.5}\right) = 27.8°$

$\theta \cong 14°$

ii) Tresca: Yielding occurs when the maximum shear stress is half the yield strength in uniaxial tension $\tau_{max} = k = \sigma_y/2$

$k = \sigma_y/2 = 300/2 = 150$ MPa

Use Mohr's circle to determine τ_{max}

τ_{max} = R = 268.4 MPa

$\tau_{max} > k$ thus yielding occurs

Von Mises: Yielding occurs when equivalent stress is equal to the yield stress, $\bar{\sigma} = \sigma_y$

$$\bar{\sigma} = \sqrt{\frac{1}{2}\left[(\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2 + 3(\sigma_{12}^2 + \sigma_{13}^2 + \sigma_{32}^2)\right]}$$

$$\bar{\sigma} = \sqrt{\frac{1}{2}\left[(400 - 0)^2 + (0 - {}^{-}75)^2 + ({}^{-}75 - 400)^2 + 3(0^2 + 125^2 + 0^2)\right]} = 492.4\,MPa$$

$\bar{\sigma} > \sigma_y$, so yielding occurs.

Problem 2 (Dislocations / Plastic Deformation)

a) List microscopic mechanisms of plastic deformation in metals, and indicate the approximate ranges over which they are dominant (express your temperatures are a fraction of the melting point, T/T_m).

b) Describe three strengthening mechanisms responsible for the hardening of metals, stating briefly how each mechanism operates (1-2 sentences).

c) An aluminum-copper alloy 2024-T81 containing a dispersion of precipitates (0.07 μm in diameter, 0.2 μm average center-to-center spacing) was tested in a standard uniaxial tension test. The specimen had a gauge length of 20 mm and a gauge diameter of 10 mm. During the test, a recording equipment malfunction allowed only the ultimate strength to be determined. Other information was obtained from handbooks and is given below:

DATA FOR 2024-T81
Shear Modulus G = 26 GPa
Poisson's Ratio $\nu = 0.31$
Ultimate tensile strength $\sigma_{UTS} = 220$ MPa
Yield strength of "pure" aluminum Al $\sigma_y = 30$ MPa
Lattice parameter a = 4Å
Coefficient of thermal expansion $\alpha = 23.6 \times 10^{-6}$ cm/cm°C

i) Estimate a lower bound for the yield strength of the alloy
ii) The rate of change of stress with respect to true strain was observed to be proportional to $\varepsilon^{-2/3}$. At what point (strain) is the specimen unstable with respect to neck formation?
iii) What load was required when the instantaneous gauge length of the specimen was 25 mm?

Solution

a) $T/T_m < 0.3$ → Twinning, slip glide
$0.3 < T/T_m < 0.7$ → Climb and glide, grain boundary sliding
$T/T_m > 0.7$ → Diffusional flow (bulk and boundary)

b) Strengthening mechanisms work by impeding dislocation motion

Cold Work: Increases the dislocation and causes interactions between dislocations. These interactions result in immobile "tangles" of dislocations which require high stress or diffusional processes to promote dislocation motion.

Solid Solutions: Atoms of differing size in solution have strain fields associated with their misfit. These atoms tend to be located near the appropriate regions around dislocations to reduce the overall strain energy. Thus, a dislocation must either "pull away" from these atoms, or carry them along in order to move.

Precipitates, Dispersions: Dislocations encountering these obstacles must pass through or around them. Cross-slip or looping is required to pass around the obstacle. To pass through, the dislocation will either want to stay within the particle (if its energy is lower inside) or not enter the particle (if its energy is higher inside.).

Grain Size: Dislocations pile up at grain boundaries. Thus a finer grain size results in more boundaries per unit area to impede dislocation motion.

c) Aluminum specimen
  i) Yield strength lower bound
  $\sigma_0 = 30$ MPa ("intrinsic yield strength of the material")

$$\Delta\tau = Gb/\ell$$

$$\ell = \bar{\ell} - 2r = 0.2\mu m - 0.07\mu m = 0.13\mu m$$

$$|b| = a/\sqrt{2} \quad \rightarrow FCC \qquad |b| = 4x10^{-10}m/\sqrt{2} = 2.8x10^{-10}m$$

$$\Delta\tau = (26x10^9\,Pa)(2.8x10^{-10}\,m)/0.13x10^{-6}\,m = 56MPa$$

$$\sigma_y = \sigma_0 + \Delta\sigma = \sigma_0 + \sqrt{3}\ \Delta\tau$$

$\sigma_y = 130$ MPa

  ii) Onset of necking

$$d\sigma/d\varepsilon \propto \varepsilon^{-2/3} \qquad d\sigma/d\varepsilon = C'\varepsilon^{-2/3}$$

Yields: $\sigma = C\varepsilon^{1/3}$ (σ=0 @ ε=0 so integration constant = 0)
At instability, dP = 0
$P = \sigma A$
$dP = \sigma dA + A d\sigma = 0$

rearranging → $d\sigma/\sigma = -dA/A = d\ell/\ell = d\varepsilon$

So, $d\sigma / d\varepsilon = \sigma$ at instability.

For $\sigma = C\varepsilon^n$, $d\sigma = nC\varepsilon^{n-1} = (n/\varepsilon)C\varepsilon^n \, d\varepsilon$

$d\sigma / d\varepsilon = \frac{n}{\varepsilon}\sigma$, but at instability, $d\sigma / d\varepsilon = \sigma$, so $\varepsilon = n = 1/3$

iii) Required load

$\ell = 2.5$ cm

$\ell_0 = 2.0$ cm

$\varepsilon = \ln(\ell/\ell_0) = \ln(2.5/2.0) = 0.223$

Solve the constitutive equation:

At UTS, $\varepsilon = n$, $\sigma = 220$ MPa

$\sigma_{UTS} = Cn^n$ → $220 \text{ MPa} = C(1/3)^{1/3}$

C = 317 MPa

@ $\varepsilon = 0.223$, $\sigma = 317(0.223)^{1/3} = 192$ MPa

$\sigma = P/A_{instant}$

To find the instantaneous area:

$d\varepsilon = d\ell/\ell = -dA/A$

$\varepsilon = -\ln A/A_0$ → $A = A_0\exp(-\varepsilon)$ $\qquad A_0 = \pi/4 \, d_0^2$

$P = \sigma A = \sigma A_0\exp(-\varepsilon)$ $\qquad d_0 = 10$ mm

$P = (195\text{x}10^6 \text{ Pa})(\pi/4)(10\text{x}10^{-3}\text{m})^2 * \exp(-0.223)$

P = 12,000 N = 12 kN

<u>Problem 3</u> (Creep)

a) List compositional and microstructural requirements for a creep resistant alloy. All other factors being equal, would you choose:

   i) A ferritic steel or an austenitic steel for service at 500°C
   ii) An alloy hardened by solid solution strengthening or precipitation hardening
   iii) A fine-grained or coarse-grained structure?

b) A chemical reaction chamber, similar to the one shown below, is made of type 304 stainless steel. The cover bolts of the chamber, which are made of the same steel, are pretensioned at room temperature to a stress of 10,000 psi. The service temperature of this chamber is 1100°F. How long can the chamber be used at this temperature before the stress in the bolts drops below 5,000 psi at the operating temperature?

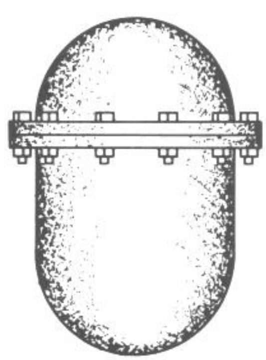
   It may be assumed that the deflection of the flanges during tightening is negligible compared to the extension of the bolts. This implies that the total strain remains constant throughout the life of the chamber.

   You may assume a constitutive law for creep for 304 stainless steel of the form:

$$\frac{\dot{\varepsilon}}{\dot{\varepsilon}_0} = \left(\frac{\sigma}{\sigma_0}\right)^m \exp\left(\frac{-H}{kT}\right)$$

   where σ_0 = 62 psi, m = 8, H = 5.85 x 10^{-18} in-lb, k = Boltzmann's constant (k = 6.79 x 10^{-23} in-lb/°R, °R = °F + 460)

   Note: The elastic modulus of the steel varies with temperature.
   $E_{RT} = 29\text{x}10^6$ psi
   $E_{1100°F} = 22.3\text{x}10^6$ psi



(Pressure vessel)

Solution:

a) Creep Deformation mechanisms:

Dislocation glide → barrier surmounted by thermal activation
Dislocation creep → barrier surmounted by diffusional processes
Diffusional creep → Coble and Nabarro-Herring (boundary vs bulk diffusion)
Grain Boundary Sliding

Obstacles which inhibit dislocation movement will also reduce the creep rate at a given temperature. Thus precipitation/dispersion hardening, and solid solution strengthening will have an effect on creep properties.

For creep purposes, grain size effects preclude use of grain boundary hardening. At low grain sizes, grain sliding is enhanced. Additionally, small grain sizes increase grain boundary diffusion rates in the Coble creep regime.

A good creep resistant alloy will have the following properties:

Solid-solution hardening
Low diffusional rates
Small stable precipitates or dispersoids
High melting point
Large grain size
Good oxidation resistance

i) Stainless steel → better oxidation resistance
ii) Precipitate → if the precipitates are stable, otherwise choose solid solution
iii) Coarse grained → both grain boundary and bulk diffusion rates increase with decreasing grain size. In addition, grain sliding increases with decreasing grain size.

b) Since the bolts and the chamber are made from the same material, the thermal expansions for the two parts re the same and the elastic strain in the bolts does not change during heating. The stress in the bolts does change, however, since the elastic modulus of the steel changes. The initial stress at 1100°F, σ_{11}^0, can be calculated from the initial stress at room temperature, in terms of the initial elastic strain, ε_{11}^0.

$$\varepsilon_{11}^0 = \frac{\sigma_{11}}{E} = \frac{10000\,psi}{29x10^6\,psi} = 3.45x10^{-4}$$

Since the strain does not change during heating, at 1100°F,

$$\sigma_{11}^0 = E\varepsilon_{11}^0 = 22.3x10^6\,psi * 3.45x10^{-4} = 7{,}700\,psi$$

Since we ignored the deflection of the flanges, the total strain remains constant during subsequent creep. As a result, the strain rate in creep is just the negative of the elastic strain rate.

$$\dot{\varepsilon}_{11}^e = -\dot{\varepsilon}_{11}^c = A\exp\left(\frac{-H}{kT}\right)(\sigma_{11})^m \text{, where } A = \dot{\varepsilon}_0(\sigma_0)^{-m} \text{ and } \sigma_{11} = E_{1100°F}\varepsilon_{11}^e$$

So, $\dot{\varepsilon}_{11}^{e} = A\exp\left(\frac{-H}{kT}\right)\left(E_{1100°F}\varepsilon_{11}^{e}\right)^{m}$

$$\int \frac{d\varepsilon_{11}^{e}}{\left(\varepsilon_{11}^{e}\right)^{m}} = \int A\exp\left(\frac{-H}{kT}\right)\left(E_{1100°F}\right)^{m} dt$$

$$\frac{\left(\varepsilon_{11}^{e}\right)^{1-m}}{m-1} = A\exp\left(\frac{-H}{kT}\right)\left(E_{1100°F}\right)^{m} t + C\,.$$

From the initial conditions that $\varepsilon_{11}^{e} = \varepsilon_{11}^{0}$ at t = 0, we get that $C = -\frac{\left(\varepsilon_{11}^{0}\right)^{1-m}}{m-1}$.

Rearranging, we get: $\frac{\left(\varepsilon_{11}^{e}\right)^{1-m} - \left(\varepsilon_{11}^{0}\right)^{1-m}}{m-1} = A\exp\left(\frac{-H}{kT}\right)\left(E_{1100°F}\right)^{m} t\,.$

Substituting back in for ε_{11}^{e} gives us

$$\left(E_{1100°F}\right)^{m-1}\frac{\left(\sigma_{11}\right)^{1-m} - \left(\sigma_{11}^{0}\right)^{1-m}}{m-1} = A\exp\left(\frac{-H}{kT}\right)\left(E_{1100°F}\right)^{m} t$$

Then we solve for t.

$$\frac{\left(E_{1100°F}\right)^{-1}}{A\exp\left(\frac{-H}{kT}\right)}\frac{\left(\sigma_{11}\right)^{1-m} - \left(\sigma_{11}^{0}\right)^{1-m}}{m-1} = t$$

We can factor out $(\sigma_{11}^{0})^{1-m}$ and rearrange to get

$$t = \frac{\left(\sigma_{11}^{0}\right)^{1-m}}{A\exp\left(\frac{-H}{kT}\right)} * \frac{1}{\left(E_{1100°F}\right)m-1} * \left[\left(\frac{\sigma_{11}^{0}}{\sigma_{11}}\right)^{1-m} - 1\right].$$

We'll plug back in for A and rearrange once last time.

$$t = \frac{\sigma_{11}^{0}}{\left(E_{1100°F}\right)(m-1)\dot{\varepsilon}_{0}\left(\sigma_{11}^{0}/\sigma_{0}\right)^{m}\exp\left(\frac{-H}{kT}\right)}\left[\left(\frac{\sigma_{11}^{0}}{\sigma_{11}}\right)^{m-1} - 1\right] \text{with } \dot{\varepsilon}_{0} = 1 \text{ hr}^{-1}.$$

Now we can plug in values and solve for t.

$$t = \frac{7700}{\left(22.3x10^{6}\right)(8-1)1\left(7700/62\right)^{8}\exp\left(\frac{-5.85x10^{-18}}{6.79x10^{-23}1560}\right)}\left[\left(\frac{7700}{5000}\right)^{8-1} - 1\right]$$

$$t = \frac{7700}{9.133}\left[20.542 - 1\right] = 1.65x10^{4}\ hr$$

Problem 4 (Fracture)

a) What is the condition we check to test for small-scale yielding? Why is checking for small-scale yielding important?

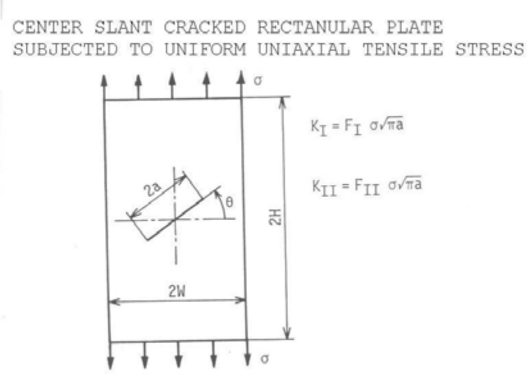
b) How does plane strain behavior arise? How is this different than the conditions necessary to exhibit plane stress behavior? What is the condition we check to test for plane strain behavior?

c) Describe the procedure for calculating fracture toughness from an R-curve. What does this value represent? Why do we care? Sketch R-curves for materials in plane strain and plane stress configurations.

d) What is a "stress intensity factor?" How is this different from a "stress concentration factor?"

e) You are designing a large plate out of 4340 steel, whose properties you can obtain from a materials handbook.

Cracks that form in this plate may not be aligned with the axis of loading. Slanted cracks may appear in the material, as shown below. If the uniform load and the length of the crack is the same, which will propagate first, a crack oriented at θ = 0°, θ = 15°, θ = 45° or θ = 60°? Why? Does your answer change if the initial crack is larger?



		θ			
	a/W	0 [1]	15 [2]	45 [2]	60 [2]
F_I	0.1	1.000	0.9391	0.5046	0.2527
	0.2	1.006	0.9577	0.5181	0.2605
	0.3	1.025	0.9904	0.5406	0.2730
	0.4	1.058	1.0402	0.5719	0.2896
	0.5	1.109	1.1128	0.6119	0.3099
	0.6	1.187	1.2183	0.6611	0.3332
	0.7	1.303	1.3780	0.7210	0.3590
	0.8	1.488	1.6530	0.7950	0.3880
F_{II}	0.1	0.0000	0.2502	0.5018	0.4352
	0.2	0.0000	0.2510	0.5072	0.4417
	0.3	0.0000	0.2527	0.5162	0.4521
	0.4	0.0000	0.2560	0.5290	0.4660
	0.5	0.0000	0.2619	0.5458	0.4827
	0.6	0.0000	0.2725	0.5674	0.5022
	0.7	0.0000	0.2900	0.5950	0.5240
	0.8	0.0000	0.3070	0.6300	0.5490

References:
[1] M. Isida, *J. Appl. Mech.,* Trans. ASME, Ser. E, 88 (1966) 674
[2] H. Kitagawa and R. Yuuki, *Trans. Japan Soc. Mech. Engrs.,* vol 43 no.376(1977) 4354

Solution:

a) To check for small scale yielding, we compare the plastic zone size to the in-plane sample dimensions, usually the crack length, a, and uncracked ligament size, b. $r_y < 1/15$ a,b. We need to check for small scale yielding if we want to use K as our fracture toughness value. We have assumed linear elastic behavior in the sample. Ahead of the crack, we developed a function based on this assumption which gives us the stress distribution. $\lim_{r \to 0} \sigma_{ij} = \frac{K}{\sqrt{2\pi r}} f_{ij}(\theta)$. This is an asymptotic solution that is most accurate where it is least relevant. That is, the stresses go to infinity as we approach the crack tip. This is nonsensical, as the material will yield once the yield strength of the material is reached. As a result, we have some region, r_y, where our assumption of linear elasticity has been violated. If this perturbed region is small, however, we can ignore its effects and continue using LEFM.

b) Plane strain behavior arises as the result of constraint ahead of the crack tip. This constraint leads to a triaxial stress state in the material ahead of the tip. These stresses counteract the Poisson's contractions caused by the applied stress. If the constraining stresses are large enough, we can hold the lateral strains = 0. This is known as plane strain behavior. It is because of this constraint (triaxial stress state) that the plastic zone size in metallic materials is smaller in plane strain than plane stress. This results in a lower K_c in plane strain than plane stress.

   We check the plastic zone size against the out-of-plane thickness of the sample. If the sample is thin enough ($r_y > 1/15$ B) then there will not be enough constraint to build up lateral stresses. The lateral stress at a free surface equals zero, so in the limit that the constraint is completely lost, we can have plane stress behavior.

   Fracture toughness, K_c is a function of crack size, thickness and geometry. As we move to thicker and thicker samples, the effects of these parameters becomes insignificant. We approach a toughness plateau known as the plane strain fracture toughness, K_{Ic}. Since K_{Ic} does not depend on thickness or geometry, we can use it as material property. If we have a thin sample ($r_y > 1/15$ B) we are not in plane strain and have to use an R-curve to determine our critical toughness value.

c) We use R-curves when the material is not in plane strain conditions. That is, there is a lack of constraint in the material which allows subcritical crack growth. We can no longer use a single-value toughness to characterize our material. The R-curve now becomes the material property for a particular part geometry and thickness.

   On an R-curve, we plot $\mathcal{G}_R$ (or K_R or J_R) versus change in crack length, Δa. As a result of local plasticity at the crack tip (intrinsic toughening) or various crack-tip shielding mechanisms (extrinsic toughening), more energy must be put into the system to continue propagating a crack. The R-curve then represents the amount of strain-energy released with crack extension.

We can also plot on this graph an applied $\mathcal{G}$ (or K or J) which represents the amount of strain-energy released with crack extension for the given applied stress. If we allow the crack to propagate by an amount da, we can then compare the applied $\mathcal{G}$ to the $\mathcal{G}_R$ required to continue propagating the crack. If the applied $\mathcal{G}$ is less than the required $\mathcal{G}$, the crack will not propagate any further. When the applied $\mathcal{G}$ is tangent to the R-curve, we have more than enough energy to continue propagating the crack and we get unstable crack growth. That is, $d\mathcal{G}/da \geq d\mathcal{G}_R/da$. The point of tangency becomes our fracture toughness value.

This value, $\mathcal{G}_c$ (or K_c or J_c) then represents the critical value beyond which a crack will grow unstably for a given geometry and thickness. Varying the initial crack length serves to move the R-curve along the x-axis of the plot.

d) Stress-concentration factor: ratio between global stress and local stress caused by geometric stress concentrations.

Stress intensity factor: characterizing parameter used in LEFM to characterize the local stresses at cracks.

e) Slanted cracks in a rectangular plate.

The crack that will propagate first is the one with the highest driving force. We'll need to convert our K_I and K_{II} values into a strain energy release rate, $\mathcal{G}$. Remember,

$\mathcal{G} = \frac{K_I^2}{E'} + \frac{K_{II}^2}{E'} + \frac{K_{III}^2}{2G}$, where E' is E in plane stress and $E/(1-\nu^2)$ in plane strain.

If we calculate the relative driving forces for each crack orientation at each a/W, we get:

	Relative Driving Force			
a/W	0°	15°	45°	60°
0.1	1	0.944509	0.506424	0.253256
0.2	1.012036	0.98019	0.525679	0.262959
0.3	1.050625	1.044749	0.558711	0.278923
0.4	1.119364	1.147552	0.606911	0.301024
0.5	1.229881	1.306915	0.672319	0.329037
0.6	1.408969	1.558511	0.758996	0.363227
0.7	1.697809	1.982984	0.873866	0.403457
0.8	2.214144	2.826658	1.028925	0.451945

As the crack gets longer, you can see that it becomes more favorable for the 15° slanted crack to propagate. However, we are assuming self-similar crack growth here. That is, we assume a planar crack will remain planar and maintain a constant shape as it grows. This is not usually the case for mixed-mode fractures. When fracture occurs, the crack will usually propagate orthogonal to the applied tensile stress. As a result, the mixed-mode crack will become a purely Mode I crack. The crack will propagate in the direction(s) that maximize the strain energy release rate, i.e. the directions with maximum driving force.

Imagine an infinitesimal kink at the end of the crack. The local stress intensities k_I and k_{II} will be a function of the global stress intensities K_I and K_{II}, as well as the kink angle, α. Figure 1 below plots the local strain-energy release rate as a function of kink angle. The peak in strain-energy release rate for each slant angle (b, which we called q in the problem) represents the point where the local Mode I stress intensity, k_I is a maximum, and the Mode II intensity, $k_{II} = 0$. You can see that each slant angle has a different optimum propagation angle.

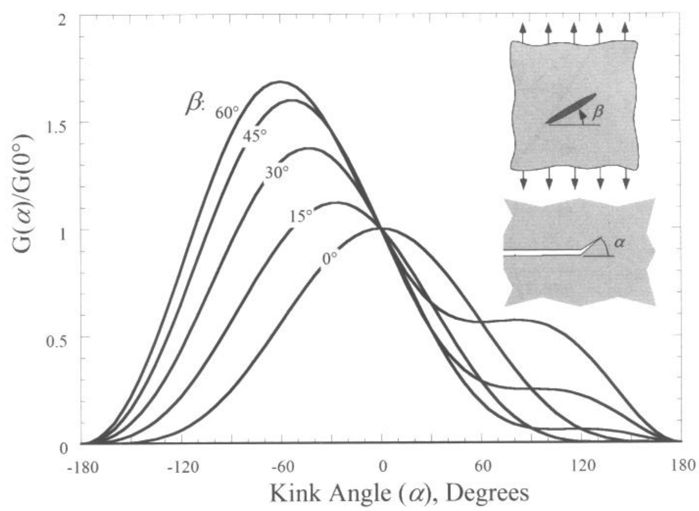


Figure 1: Local strain-energy release rate at the tip of a kinked crack.
From *Fracture Mechanics* 3 ed.by T. L. Anderson

<u>Problem 5</u> (Fracture and Fatigue)

A coal gasification pressure vessel is to be fabricated from an advanced low-alloy 3Cr-1Mo steel. Power generation requirements dictate that the vessel be 14 ft in diameter and 41 ft in length. The vessel is to operate at a mean pressure and temperature of 2500 psi and 600°F. Safety considerations require that the vessel maintain its integrity at <u>both</u> the operating temperature and room temperature. Relevant data for the steel are provided below.

PROPERTIES OF LOW-ALLOY STEEL

T (°F)	σ_y (ksi)	σ_{UTS} (ksi)	K_{Ic} (ksi$\sqrt{in}$)
60	50	100	190
600	30	60	230

Fatigue Law: $da/dN = C\,\Delta K^3$, where $C = 1\text{x}10^{-12}$ (units in ksi$\sqrt{in}$ and in/cycle)

If you are required to use linear-elastic fracture mechanics:

a) Calculate the required plate thickness, t, such that the vessel will not yield and will leak before it breaks (i.e. $a_f \leq t$). Use the Von Mises criterion to check for yielding. Assume $K=\sigma\sqrt{\pi a}$.

b) Operational considerations result in a pressure fluctuation of ±10% of the mean pressure at a frequency of 0.5 Hz. Assuming that there are no flaws present initially greater than 2 inches (well above detection limits), calculate the wall thickness required such that the vessel can operate safely for 30 years.

c) Embrittlement of the material due to radiation enhanced precipitation of impurity elements results a loss of toughness with operating time:

$$K_{Ic}(\tau) = K_0 - \alpha\tau$$

where K_0 = initial toughness
α = 5 ksi$\sqrt{in}$ per year
τ = operating time (in years)

Assuming that the thickness of the vessel is that calculated in part (b), how long can the vessel be operated safely?

d) We have assumed linear elastic fracture mechanics. How applicable is that assumption for this example?

Solution:

a) Leak before break criterion:

$a_f = t$

$K = \sigma\sqrt{\pi a}$, and the max stress is given by $\sigma_{\theta\theta} = \frac{\Pr}{t}$.

K_{Ic}^{min} is K_{Ic} at room temperature (190 ksi$\sqrt{in}$), so $K_{max} = K_{Ic}^{min}$ is our criterion.

$$K_{\max} = \frac{\Pr}{t}\sqrt{\pi t} = 190x10^3\, psi\sqrt{in}$$

$$K_{\max} = \frac{2500\,psi * 84in * \sqrt{\pi}}{\sqrt{t}} = 190x10^3\, psi\sqrt{in}$$

$t \geq 3.8"$

Yielding:

$$\bar{\sigma} = \sqrt{\frac{1}{2}\left[(\sigma_{\theta\theta} - \sigma_{zz})^2 + (\sigma_{zz} - \sigma_{rr})^2 + (\sigma_{rr} - \sigma_{\theta\theta})^2\right]}$$, but $\sigma_{\theta\theta} = 2\sigma_{zz}$, so

$$\bar{\sigma} = \sqrt{\frac{1}{2}\left[(2\sigma_{zz} - \sigma_{zz})^2 + (\sigma_{zz} - 0)^2 + (0 - 2\sigma_{zz})^2\right]} = \sqrt{\frac{1}{2}\left[(\sigma_{zz})^2 + (\sigma_{zz})^2 + (-2\sigma_{zz})^2\right]} = \sqrt{3}\sigma_{zz}$$

$$\sqrt{3}\sigma_{zz} = \sqrt{3}\frac{\Pr}{2t}.$$

The limiting case for yielding is σ_y @ T = 600°F.

$$30{,}000\,psi = \sqrt{3}\frac{2500\,psi * 84in}{2t}$$

$t \geq 6.1"$. This is our required thickness to satisfy both requirements.

b) $\sigma_{\theta\theta} = \frac{\Pr}{t} = \sigma_{max.}$

$\Delta K = \Delta\sigma\sqrt{\pi a} = \frac{\Delta \Pr}{t}\sqrt{\pi a}$, where $\Delta P = 500\,psi$.

$\frac{da}{dN} = C\Delta K^3 = C\left(\Delta\sigma\sqrt{\pi a}\right)^3 = C\left(\frac{\Delta \Pr}{t}\sqrt{\pi a}\right)^3$, where C = $1x10^{-12}$ in/cycle.

$\frac{da}{d\tau} = \nu C\Delta K^3 = \nu C\left(\Delta\sigma\sqrt{\pi a}\right)^3 = \nu C\left(\frac{\Delta \Pr}{t}\sqrt{\pi a}\right)^3$, where ν = 0.5 cycle/second.

$$da = \nu C\left(\frac{\Delta \Pr}{t}\sqrt{\pi}\right)^3 a^{\frac{3}{2}} d\tau$$

$\frac{da}{a^{\frac{3}{2}}} = \nu C\left(\frac{\Delta \Pr}{t}\sqrt{\pi}\right)^3 d\tau = C' d\tau$, where $C' = \nu C\left(\frac{\Delta \Pr}{t}\sqrt{\pi}\right)^3$.

$\int_{a_0}^{a_f} \frac{da}{a^{\frac{3}{2}}} = \int_0^{\tau_f} C' d\tau$, where t_f = 30 years = $9.46x10^8$ seconds.

$a_0 = 2"$, $a_f = t$. $\int \frac{da}{a^{\frac{3}{2}}} = -2a^{-1/2}$, so

$$\frac{2}{\sqrt{a_0}} - \frac{2}{\sqrt{a_f}} = \frac{2}{\sqrt{a_0}} - \frac{2}{\sqrt{t}} = \nu C\left(\frac{\Delta \Pr}{t}\sqrt{\pi}\right)^3 \tau_f$$

$$\frac{2}{\sqrt{2"}} - \frac{2}{\sqrt{t}} = 0.5 * 1x10^{-12}\left(\frac{500 psi * 84in}{t}\sqrt{\pi}\right)^3 9.46x10^8$$

$1.414 - 2t^{-1/2} = 195t^{-3}$.

Solving for t, we get $t \geq 6.7"$

c) Again, the criterion for fracture is $K_{max} = K_{Ic}^{min}$.

$K_{max}(\tau) = K_{Ic}^{min} - \alpha\tau = K_0 - \alpha\tau$, where α = 5 ksi $\sqrt{in}$ / year and K_0 = 190 ksi $\sqrt{in}$.

$K_{max} = \sigma_{max}\sqrt{\pi a(\tau)}$, where $\sigma_{max} = \frac{P_{max} r}{t} = \frac{2750 psi * 84"}{6.7"} = 34.5 ksi$.

Again, $\frac{da}{d\tau} = \nu C \Delta K^3 = \nu C\left(\Delta\sigma\sqrt{\pi a}\right)^3 = \nu C\left(\frac{\Delta \Pr}{t}\sqrt{\pi a}\right)^3$. From b) we get:

$\frac{2}{\sqrt{a_0}} - \frac{2}{\sqrt{a}} = \nu C\left(\frac{\Delta \Pr}{t}\sqrt{\pi}\right)^3 \tau_f$ which rearranges to:

$$\frac{2}{\sqrt{a}} = \frac{2}{\sqrt{a_0}} - \nu C\left(\frac{\Delta \Pr}{t}\sqrt{\pi}\right)^3 \tau_f \rightarrow a^{\frac{-1}{2}} = a_0^{\frac{-1}{2}} - \frac{\nu C}{2}\left(\frac{\Delta \Pr}{t}\sqrt{\pi}\right)^3 \tau_f$$

$$a(\tau) = \left\{a_0^{\frac{-1}{2}} - \frac{\nu C}{2}\left(\frac{\Delta \Pr}{t}\sqrt{\pi}\right)^3 \tau_f\right\}^{-2} = \left\{0.707 - \frac{0.5 * 1x10^{-12}}{2}\left(\frac{500 psi * 84"}{6.7"}\sqrt{\pi}\right)^3 \tau_f\right\}^{-2}$$

$a(\tau) = \{0.707 - 3.4x10^{-10}\tau_f\}^{-2}$ with τ_f in seconds. If we convert to years, we get:

$a(\tau) = \{0.707 - 0.011\tau_f\}^{-2}$ since 1 year = $31.56 x 10^6$ seconds.

Ok, so now we get $K_{Ic} = K_0 - \alpha\tau = K_{max}(\tau) = \frac{P_{max} r}{t}\sqrt{\pi a(\tau)}$

Plugging in, we get $K_0 - \alpha\tau_f = \frac{P_{max} r}{t}\sqrt{\pi\{0.707 - 0.011\tau_f\}^{-2}}$

$$190 ksi\sqrt{in} - 5\frac{ksi\sqrt{in}}{year}\tau_f = \frac{2750 psi * 84"}{6.7"}\sqrt{\pi\{0.707 - 0.011\tau_f\}^{-2}}$$

$$190 - 5\tau_f = \frac{61}{\{0.707 - 0.011\tau_f\}}.$$

Solving for τ_f, we get τ_f = 15 years.

d) The criteria for LEFM is $B \geq 2.5\left(\frac{K_{Ic}}{\sigma_y}\right)^2$ and $b \geq 15r_y \approx \frac{15}{2\pi}\left(\frac{K_{Ic}}{\sigma_y}\right)^2$.

Initially, K_{Ic}^{max} = 230 ksi$\sqrt{in}$ and σ_y = 30 ksi at 600°F.

$b \geq \frac{15}{2\pi}\left(\frac{230}{30}\right)^2 \rightarrow b \geq 150"$.

LEFM is not valid in this analysis. Methods using J_{Ic} should be employed.