Math 104 - Midterm 1 Solutions
Lecture 4, Fall 2011

1. (10 points) Give an example of each of the following. You do not need to give any justification.
(a) A nonempty, bounded subset of $\mathbb{R}\backslash\mathbb{Q}$ with no infimum in $\mathbb{R}\backslash\mathbb{Q}$.
(b) A subspace of $\mathbb{R}$ containing $\mathbb{Z}$ in which $\{-4\}$ is open but $\{3\}$ is not.
(c) A nonempty subspace of $\mathbb{R}\backslash\mathbb{Q}$ which is complete in the metric space sense.
(d) An uncountable open subset of $(-1,1)\cap(\mathbb{R}\backslash\mathbb{Q})$ which is not all of $(-1,1)\cap(\mathbb{R}\backslash\mathbb{Q})$.
(e) A bounded sequence in $\{x\in\mathbb{R}\backslash\mathbb{Q} \mid -\pi<x<\pi\}$ which is not Cauchy in $\mathbb{R}$.

Solution. Here are some possible examples. Also, the problem did not require justifications but I'll give some anyway.

(a) The set $(-2,2)\cap\mathbb{R}\backslash\mathbb{Q}$ of irrational numbers between -2 and 2 works. This is certainly bounded since it sits inside of the bounded interval $(-2,2)$. Also, as a subset of $\mathbb{R}$ its infimum is -2, which is not irrational. Hence the given set has no infimum in $\mathbb{R}\backslash\mathbb{Q}$.

(b) The set $S=\mathbb{Z}\cup\left(\frac{5}{2},\frac{7}{2}\right)$ works. This certainly contains $\mathbb{Z}$. Also, $\{-4\}$ is open in S since the ball of radius $\frac{1}{4}$ in S around -4 only contains -4, so this ball is contained in $\{-4\}$. (The point is that there are no other elements of S in this ball.) However, $\{3\}$ is not open in S since any ball in S around 3 will contain elements of S which are not in $\{3\}$.

(c) The sets $\sqrt{2}\mathbb{N}=\{\sqrt{2}n \mid n\in\mathbb{N}\}$ and $\sqrt{2}\mathbb{Z}=\{\sqrt{2}m \mid m\in\mathbb{Z}\}$ work. In both of these, the only Cauchy sequences are ones which are eventually constant since the distance between distinct points of these can never be smaller than 1. Actually, by a similar reasoning, any nonempty finite subset of $\mathbb{Q}$ will also be such an example.

(d) The set $\{x\in\mathbb{R}\backslash\mathbb{Q} \mid 0<x<1\}$ of positive irrational numbers works less than 1 works. Indeed, this is uncountable and for any such x the ball of radius $\min\{|x|,1-|x|\}$ in $\mathbb{R}\backslash\mathbb{Q}$ consists only of positive irrational numbers less than 1, so this ball is contained in the above set.

(e) The sequence $(-1)^n\sqrt{2}$ works. The terms of this sequence are all in the given set, which is itself bounded, and this sequence does not converge since it has a subsequence converging to $-\sqrt{2}$ and another converging to $\sqrt{2}$. Since it does not converge in $\mathbb{R}$, it is not Cauchy in $\mathbb{R}$. □

2. (15 points) For each $n\in\mathbb{N}$, let $f_n:[0,1]\to\mathbb{R}$ denote the function defined by

$$f_n(x):=\begin{cases}\frac{1}{3}x^n & \text{if } x\in\mathbb{R}\backslash\mathbb{Q}\\ 0 & \text{otherwise.}\end{cases}$$

Compute the distance between $10f_n$ and f_n in $C_b([0,1])$ with respect to the sup metric and prove that your answer is correct. (Careful: note that $f_n(1)=0$ and not 1 for all n.)

Solution. The distance we want is

$$\sup_{x\in[0,1]}|10f_n(x)-f_n(x)|=\sup_{x\in[0,1]}9|f_n(x)|.$$

For rational x, $f_n(x)=0$ so such x cannot possibly give the supremum we want. So, the above supremum comes from considering irrational x:

$$\sup_{x\in[0,1]}9|f_n(x)|=\sup_{x\in[0,1]\cap(\mathbb{R}\backslash\mathbb{Q})}3|x|^n.$$

As x gets closer to 1, $3|x|^n$ gets closer to 3 so we claim that the supremum is indeed 3.

First, since $x^n \leq 1$ for any $0 \leq x \leq 1$, we see that 3 is an upper bound of $\{3|x|^n \mid x \in (\mathbb{R}\backslash\mathbb{Q}) \cap [0,1]\}$. Now, let $\epsilon > 0$ and $\epsilon \leq 3$. By the denseness of $\mathbb{R}\backslash\mathbb{Q}$ in $\mathbb{R}$, there exists an irrational number x such that

$$\sqrt[n]{\frac{3-\epsilon}{3}} < x < 1.$$

Note that the condition $\epsilon \leq 3$ is required in order to guarantee that the n-th root above is defined. Then

$$\frac{3-\epsilon}{3} < x^n < 1, \text{ so } 3-\epsilon < 3x^n < 3.$$

Thus we have found an element of $\{3|x|^n \mid x \in (\mathbb{R}\backslash\mathbb{Q}) \cap [0,1]\}$ which is larger than $3-\epsilon$. For $\epsilon > 3$, any element of this set will be larger than $3-\epsilon$ since in this case $3-\epsilon < 0$. Thus 3 satisfies the ϵ-characterization of supremums, so we conclude that $d(10f_n, f_n) = 3$ for any n as claimed. □

3. (15 points) Suppose that (x_n) is a sequence of real numbers and that $a, b \in \mathbb{R}$ with $a \neq 0$.

(a) If (x_n) converges to x and $ax_n + b \geq 0$ for all n, show that $(\sqrt{ax_n+b})$ converges to $\sqrt{ax+b}$.

(b) Give an example, with brief justification, where (x_n^4) converges but (x_n) does not.

(c) If (x_n^2) converges to 0, show that (x_n) converges to 0.

In (a) you must use only the definition of convergence and no other limit theorems.

Solution. (a) First note that the condition $ax_n + b \geq 0$ is only there to ensure that the square roots we use all make sense. Let $\epsilon > 0$. Since $(x_n) \to x$, we can pick an index N such that

$$|x_n - x| < \frac{\epsilon^2}{\sqrt{|a|}} \text{ for } n \geq N.$$

Using the fact that $\sqrt{\alpha} - \sqrt{\beta} \leq \sqrt{\alpha - \beta}$ whenever $\alpha \geq \beta$, we have

$$|\sqrt{ax_n+b} - \sqrt{ax+b}| \leq \sqrt{|(ax_n+b)-(ax+b)|},$$

so for $n \geq N$ we have

$$|\sqrt{ax_n+b} - \sqrt{ax+b}| \leq \sqrt{|ax_n - ax|} = \sqrt{|a|}\sqrt{|x_n - x|} < \sqrt{\epsilon^2} = \epsilon.$$

Thus $(\sqrt{ax_n+b}) \to \sqrt{ax+b}$ as claimed.

(b) The sequence $x_n = (-1)^n$ works. Indeed, we have $x_n^4 = 1$ is a constant sequence so it converges, but (x_n) does not converge since the subsequence of even-indexed terms converges to 1 while the subsequence of odd-indexed terms converges to -1.

(c) Let $\epsilon > 0$ and pick N such that

$$|x_n|^2 = |x_n^2 - 0| < \epsilon^2 \text{ for } n \geq N.$$

Taking square roots we have

$$|x_n - 0| = |x_n| < \epsilon \text{ for } n \geq N,$$

showing that (x_n) converges to 0. □

4. (10 points) Suppose that (X, d) is a metric space and let $\{x_1, \ldots, x_n\}$ be a finite set of points of X. Show, using only the definition of open, that the set $X\backslash\{x_1, \ldots, x_n\}$ obtained by removing each x_i from X is open in X. (Draw a picture to get some intuition!)

Proof. Let $p \in X \backslash \{x_1, \ldots, x_n\}$ and set

$$r = \min\{d(x_1, p), \ldots, d(x_n, p)\}.$$

Since $p \neq x_i$ for all i, each of the distances above is positive so their minimum r is also positive. If $q \in B_r(p)$, we have

$$d(q, x_i) \geq d(x_i, p) - d(q, p) > d(x_i, p) - r \geq 0 \text{ for all } i = 1, \ldots, n,$$

so $q \neq x_i$ for any i. Thus $B_r(p) \subseteq X \backslash \{x_1, \ldots, x_n\}$ and $X \backslash \{x_1, \ldots, x_n\}$ is open in X as claimed. □

5. (0 points) Draw a picture of your favorite closed subset of $\mathbb{R}^2$.