Math 54 Midterm 2 Solution

(1) One version: FFTTTTFF FTFTFTFT TTTTFFFF
Another version: TFTFTFTF FFFTTTFT FTFTFTFT
Yet another version: TTFFFFTT TTTFFFTF FFFFTTTT

(2) (b) The eigenvalues of A are 6 and 1. A 6-eigenvector is $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. A 1-eigenvector is $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$. So

$$S = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}, \qquad D = \begin{bmatrix} 6 & 0 \\ 0 & 1 \end{bmatrix}$$

(c) The sequence of matrices $\lambda^{-n}D^n$ has a nonzero limit if and only if both of the sequences of diagonal entries $\lambda^{-n}6^n$ and λ^{-n} have a limit and at least one of these limits is nonzero. The only value of λ for which this happens is $\lambda = 6$: if $\lambda < 0$ or if $0 < \lambda < 1$, neither converges; if $1 \le \lambda < 6$, $\lambda^{-n}6^n$ does not converge; and if $\lambda > 6$, they both converge to zero. Whereas if $\lambda = 6$, $\lambda^{-n}6^n = 1$ for all n, and $\lambda^{-n}1^n \to 0$, so $\lambda^{-n}D^n \to \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Since $A = SDS^{-1}$, $A^n = SD^nS^{-1}$, so $\lambda^{-n}A^n$ converges if and only if $\lambda^{-n}D^n$ converges. The limit in this case is

$$S\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}S^{-1} = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1/5 & 2/5 \\ 2/5 & -1/5 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1/5 & 2/5 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1/5 & 2/5 \\ 2/5 & 4/5 \end{bmatrix}$$

(3) (a) Let $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$, and let W denote the plane they span in $\mathbb{R}^4$. Construct an orthogonal basis for W by setting $\mathbf{u}_1 = \mathbf{v}_1$ and $\mathbf{u}_2 = \mathbf{v}_2 - \frac{\mathbf{v}_2\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} - \frac{6}{9}\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2/3 \\ -1/3 \\ 2/3 \end{bmatrix}$. Now normalize $\mathbf{u}_1$ and $\mathbf{u}_2$ to give the orthonormal basis consisting of $\mathbf{q}_1 = \begin{bmatrix} -1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$ and $\mathbf{q}_2 = \mathbf{u}_2 = \begin{bmatrix} 2/3 \\ -1/3 \\ 2/3 \end{bmatrix}$ ($\mathbf{u}_2$ is already a unit vector).

(b) A vector $\mathbf{q}_3$ orthogonal to both $\mathbf{q}_1$ and $\mathbf{q}_3$ must lie in the orthogonal complement of their span, which is one dimensional; in a one-dimensional space, there are only two unit vectors, so there will be two choices for $\mathbf{q}_3$. The orthogonal complement to the space spanned by $\mathbf{q}_1$ and $\mathbf{q}_2$ is the nullspace of the matrix having $\mathbf{q}_1$ and $\mathbf{q}_2$ as rows:

$$\text{Null}\begin{bmatrix} -1/3 & 2/3 & 2/3 \\ 2/3 & -1/3 & 2/3 \end{bmatrix} \overset{\text{row ops}}{=} \text{Null}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

This null space is one-dimensional, spanned by $\begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}$. This vector is orthogonal to $\mathbf{q}_1$ and $\mathbf{q}_2$, and has length 3. So the two possibilites for $\mathbf{q}_3$ are $\pm\begin{bmatrix} 2/3 \\ 2/3 \\ -1/3 \end{bmatrix}$. Either choice of $\mathbf{q}_3$ gives a matrix $Q = [\mathbf{q}_1\ \mathbf{q}_2\ \mathbf{q}_3]$ which is orthogonal. Fix the choice with $\mathbf{q}_3 = \begin{bmatrix} 2/3 \\ 2/3 \\ -1/3 \end{bmatrix}$.

(c) The matrix Q is

$$Q = \begin{bmatrix} -1/3 & 2/3 & 2/3 \\ 2/3 & -1/3 & 2/3 \\ 2/3 & 2/3 & -1/3 \end{bmatrix}$$

Since Q is symmetric, its eigenvalues are real, and since it's orthogonal, its eigenvalues have length one, so the eigenvalues of Q are either ± 1. First look at $\lambda = 1$:

$$Q - \lambda I = \begin{bmatrix} -4/3 & 2/3 & 2/3 \\ 2/3 & -4/3 & 2/3 \\ 2/3 & 2/3 & -4/3 \end{bmatrix} \overset{\text{row ops}}{\sim} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

So the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ spans the 1-eigenspace. On the other hand, when $\lambda = -1$, we get

$$Q - \lambda I = \begin{bmatrix} 2/3 & 2/3 & 2/3 \\ 2/3 & 2/3 & 2/3 \\ 2/3 & 2/3 & 2/3 \end{bmatrix} \overset{\text{row ops}}{\sim} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

so the -1-eigenspace is spanned by $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

If we chose instead the other possibility for $\mathbf{q}_3$, then we would have

$$Q = \begin{bmatrix} -1/3 & 2/3 & -2/3 \\ 2/3 & -1/3 & -2/3 \\ 2/3 & 2/3 & 1/3 \end{bmatrix}$$

The eigenvalues of this matrix are -1 and $\frac{1}{3}(1 \pm 2\sqrt{2}i)$. Choosing $\lambda = -1$, we compute the eigenspace as the nullspace of

$$\begin{bmatrix} 2/3 & 2/3 & -2/3 \\ 2/3 & 2/3 & -2/3 \\ 2/3 & 2/3 & 4/3 \end{bmatrix} \overset{\text{row ops}}{\sim} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

This nullspace is spanned by $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, so this is an eigenvector for $\lambda = -1$.

(4) We must solve the system of equations $y_i = ax_i^3 + bx_i^2$, for $i = 1, 2, 3, 4$. This corresponds to the linear system $A\mathbf{x} = \mathbf{y}$, where

$$A = \begin{bmatrix} -1 & 1 \\ 0 & 0 \\ 1 & 1 \\ 8 & 4 \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} a \\ b \end{bmatrix}, \qquad \mathbf{y} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 4 \end{bmatrix}$$

This system is inconsistent, but the least squares solution $\widehat{\mathbf{x}}$ can be obtained by solving the normal equation

$$A^T A \widehat{\mathbf{x}} = A^T \mathbf{y}$$

Here $A^T A = \begin{bmatrix} 66 & 32 \\ 32 & 18 \end{bmatrix}$ and $A^T \mathbf{y} = \begin{bmatrix} 33 \\ 19 \end{bmatrix}$, so the augmented matrix for the normal equation is

$$\begin{bmatrix} 66 & 32 & 33 \\ 32 & 18 & 19 \end{bmatrix} \overset{\text{row ops}}{\sim} \begin{bmatrix} 1 & 0 & -7/82 \\ 0 & 1 & 99/82 \end{bmatrix}$$

So the desired equation is

$$y = -\frac{7}{82}x^3 + \frac{99}{82}x^2.$$