Math 54 Final Exam Solution

(1) Multiple Choice
(1) b (2) d (3) d (4) b (5) a (6) d (7) d (8) XXX (9) b (10) c
(11) d (12) b (13) c (14) c (15) b (16) b (17) b (18) b (19) b (20) b

(2) (a) Describe Lagrange's method of "Variation of parameters' for solving the second-order inhomogeneous ODE

$$x''(t) + a_1(t)x'(t) + a_0(t)x(t) = g(t)$$

(b) Solve the ODE $\frac{1}{2}x'' + 2x = \tan(2t)$, $-\frac{\pi}{4} < x < \frac{\pi}{4}$.

Solution: First solve the homogeneous equation $\frac{1}{2}x'' + 2x = 0$, whose characteristic equation $\frac{1}{2}r^2 + 2 = 0$ has roots $r = \pm 2i$, giving us the two independent (real) solutions

$$x_1(t) = \cos 2t, \quad x_2(t) = \sin 2t$$

We now need one particular solution x_p to the original equation. This can be done by variation of parameters, setting $x_p = v_1 x_1 + v_2 x_2$, where v_1 and v_2 are to be determined. They are found by solving the equations

$$\begin{aligned} x_1 v_1' + x_2 v_2' &= 0 \\ x_1' v_1' + x_2' v_2' &= 2\tan(2t), \end{aligned}$$

where the factor of 2 on $\tan 2t$ is a result of the coefficient $\frac{1}{2}$ on x''. This is equivalent to the matrix equation

$$\begin{bmatrix} \cos 2t & \sin 2t \\ -2\sin 2t & 2\cos 2t \end{bmatrix} \begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \begin{bmatrix} 0 \\ 2\tan 2t \end{bmatrix}.$$

The (Wronskian) matrix on the left is invertible since x_1 and x_2 are independent, so we can invert it and compute

$$\begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \begin{bmatrix} \cos 2t & \sin 2t \\ -2\sin 2t & 2\cos 2t \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 2\tan 2t \end{bmatrix} = \begin{bmatrix} \cos 2t & -\frac{1}{2}\sin 2t \\ \sin 2t & \frac{1}{2}\cos 2t \end{bmatrix} \begin{bmatrix} 0 \\ 2\tan 2t \end{bmatrix} = \begin{bmatrix} -\sin 2t \tan 2t \\ \sin 2t \end{bmatrix}$$

To find v_1 we must find an antiderivative of $-\sin 2t \tan 2t$, as follows:

$$\begin{aligned} -\int \sin 2t \tan 2t \, dt &= -\int \frac{\sin^2 2t}{\cos 2t} dt = -\int \frac{1 - \cos^2 2t}{\cos 2t} dt \\ &= -\int \frac{1}{\cos 2t} dt + \int \cos 2t dt \\ &= -\frac{1}{2} \log \frac{\sin t + \cos t}{\sin t - \cos t} + \frac{1}{2} \sin 2t \end{aligned}$$

Thus $v_1 = -\frac{1}{2} \log \frac{\sin t + \cos t}{\sin t - \cos t} + \frac{1}{2}\sin 2t$. Also, $v_2 = \int \sin 2t dt = -\frac{1}{2}\cos 2t$. Putting this all together, we find that

$$x_p = \cos 2t \left(-\frac{1}{2} \log \frac{\sin t + \cos t}{\sin t - \cos t} + \frac{1}{2} \sin 2t \right) + \sin 2t \left(-\frac{1}{2} \cos 2t \right) = -\frac{1}{2} \cos 2t \log \frac{\sin t + \cos t}{\sin t - \cos t},$$

so the general solution is

$$x(t) = c_1 \cos 2t + c_2 \sin 2t - \frac{1}{2} \cos 2t \log \frac{\sin t + \cos t}{\sin t - \cos t}$$

(3) (a) Find the Fourier cosine series for the function $f(x) = \sin x$ on the interval $[0, \pi]$.
(b) Specialize your Fourier series to $x = \pi/2$ to get an interesting identity.
Solution:
(a) The coefficients a_n ($n \geq 0$) for the cosine series are given by the formulae

$$a_n = \frac{2}{\pi} \int_0^\pi \sin x \cos nx \, dx$$

First notice $a_0 = \frac{2}{\pi} \int_0^\pi \sin x dx = \frac{4}{\pi}$. To compute the integral for $n > 0$, recall that $\sin(A + B) = \sin A \cos B + \sin B \cos A$ and $\sin(A - B) = \sin A \cos B - \sin B \cos A$. Subtracting these formulae gives the relation $\sin(A + B) - \sin(A - B) = 2 \sin B \cos A$. We use this in the above integral, taking $B = x$, $A = nx$. This gives

$$a_n = \frac{2}{\pi} \int_0^\pi \frac{1}{2} \left(\sin(n+1)x - \sin(n-1)x \right) dx$$

When $n = 1$, the integrand is $\sin 2x - \sin 0 = \sin 2x$, so $a_1 = \frac{2}{\pi} \int_0^\pi \frac{1}{2} \sin 2x dx = \frac{2}{\pi}\frac{1}{2} \left[-\frac{1}{2} \cos 2x \right]_0^\pi = 0$. For $n \geq 2$, we calculate

$$
\begin{aligned}
a_n &= \frac{2}{\pi}\int_0^\pi \frac{1}{2}\left(\sin(n+1)x - \sin(n-1)x\right)dx \\
&= \frac{1}{\pi}\left[-\frac{1}{n+1}\cos(n+1)x + \frac{1}{n-1}\cos(n-1)x\right]_0^\pi \quad \text{(this makes sense since } n \neq 1\text{)} \\
&= \frac{1}{\pi}\left[-\frac{1}{n+1}(-1)^{n+1} + \frac{1}{n-1}(-1)^{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right] \\
&= \frac{1}{\pi}\left[(-1)^n\left(\frac{1}{n+1} - \frac{1}{n-1}\right) + \frac{1}{n+1} - \frac{1}{n-1}\right] \\
&= \frac{1}{\pi}\left((-1)^n + 1\right)\left(\frac{1}{n+1} - \frac{1}{n-1}\right) \\
&= \frac{1}{\pi}\left((-1)^n + 1\right)\left(\frac{-2}{n^2-1}\right) = \begin{cases} \frac{4}{\pi(1-n^2)} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}
\end{aligned}
$$

Thus the fourier cosine series for f is

$$\sin x \sim \frac{2}{\pi} + \sum_{n=2}^{\infty}\frac{1}{\pi}\left((-1)^n+1\right)\left(\frac{4}{1-n^2}\right)\cos nx = \frac{2}{\pi} + \sum_{k=1}^{\infty}\frac{4}{\pi(1-4k^2)}\cos 2kx.$$

(b) At $x = \pi/2$, we get after multiplying through by π, $\pi = 2 + \sum_{k=1}^{\infty}\frac{(-1)^k 4}{1-4k^2}$.

(4) (a) Write the general form of d'Alembert's solution to the wave equation $\dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 u}{\partial x^2}$ for a function $u(x,t)$ with $x, t \in \mathbb{R}$. Briefly explain how the ingredients of the solution can be found from the initial conditions.
(b) By d'Alembert's method or otherwise, solve the equation explicitly with 2π-periodic functions of x, subject to the intial conditions $u(x,0) = \sin^2 x$ and $\frac{\partial u}{\partial t}(x,0) = \cos x$.

Solution:
(a)
(b) Using d'Alembert's formula

$$
\begin{aligned}
u(x,t) &= \frac{1}{2}\left(\sin^2(x+t) + \sin^2(x-t)\right) + \frac{1}{2}\left(\sin(x+t) - \sin(x-t)\right) \\
&= \frac{1}{2}\left(\sin^2(x+t) + \sin^2(x-t) + 2\sin t\cos x\right) \\
&= (\sin x\cos t)^2 + (\sin t\cos x)^2 + \sin t\cos x
\end{aligned}
$$

(5) (a) Write the general solution of the vector-valued ODE

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} 1.4 & 1.6 \\ -.8 & -.2 \end{bmatrix}\mathbf{x}.$$

(b) Draw a "phase diagram" of this ODE, roughly sketching a few trajectories.
(c) Which axis is the first to be crossed, if we start with the initial value $\mathbf{x}(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$?

Solution:
(a) Let A denote the matrix appearing in the equation. The solutions to a first-order vector ODE have the form $e^{\lambda t}\mathbf{x}$, where $\mathbf{x}$ is an eigenvector for A with eigenvalue λ. The eigenvalues of A are $\lambda = \frac{3}{5} \pm \frac{4}{5}i$. An eigenvector for $\lambda = \frac{3}{5} + \frac{4}{5}i$ is $\mathbf{v} = \begin{bmatrix} 1+i \\ -1 \end{bmatrix}$. This breaks into real and imaginary parts $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, yielding two real independent solutions

$$\mathbf{x}_1(t) = e^{\frac{3}{5}t}\left(\cos\frac{4}{5}t\begin{bmatrix} 1 \\ -1 \end{bmatrix} - \sin\frac{4}{5}t\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) \quad \text{and} \quad \mathbf{x}_2(t) = e^{\frac{3}{5}t}\left(\sin\frac{4}{5}t\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \cos\frac{4}{5}t\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$

Then the general solution has the form $\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t)$, where $c_1, c_2 \in \mathbb{R}$ are arbitrary.
(b)
(c) The condition $\mathbf{x}(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ allows us to solve for c_1 and c_2. Notice that at $t = 0$, the exponentials are all 1, and the terms involving $\sin\frac{4}{5}t$ vanish, so $\mathbf{x}(0) = c_1\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and we see by inspection or by writing out a system of two equations in the unknowns c_1, c_2 that $c_1 = -1$ and $c_2 = 3$. So the desired solution is

$\mathbf{x}(t) = -\mathbf{x}_1(t) + 3\mathbf{x}_2(t)$, where $\mathbf{x}_1$ and $\mathbf{x}_2$ are as above. Simplifying this yields

$$\mathbf{x}(t) = e^{\frac{3}{5}t}\left(\cos\frac{4}{5}t\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \sin\frac{4}{5}t\begin{bmatrix} 4 \\ -3 \end{bmatrix}\right)$$

The trajectory of this solution is an "outward" spiral which begins ($t = 0$) at $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and moves initially toward the vector $\begin{bmatrix} 4 \\ -3 \end{bmatrix}$. Thus the first axis to be crossed is the $x$axis.