Mathematics 1A, Fall 2009 — M. Christ

Midterm Exam #1

NAME:

Your CSI's NAME and your DISCUSSION SECTION time:

Sec 103, MW 9-10am

Instructions:

1. Closed book exam — No formula sheets or notes are permitted. Calculating and other electronic devices are not allowed. Turn cell phones off and stow them in backpacks/pockets/purses.
2. Do all work in this booklet. Use the blank sheet provided on the last page, or use backs of pages, for scratch work.
3. **Show work and/or reasoning where indicated.**

Problem	Point Value		
1	$10 = 7 + 3$	10	
2	$12 = 6 + 6$	9	
3	6	6	
4	$10 = 5 \times 2$	4+5	9
5	$12 = 6 + 6$	5+7	12
Total	50	46	

There are 5 problems on this exam, with 12 parts in all. All problems have short solutions, except (5b). Work efficiently. If you don't know how to attack a problem, go on and come back to it later.

(1a) *7 points.* Use limit rules to evaluate $\lim_{x\to 9} \frac{\sqrt{x}-3}{x-9}$. Show your steps.

$$\lim_{x\to 9} \frac{\sqrt{x}-3}{x-9}$$

$$= \lim_{x\to 9} \frac{\cancel{\sqrt{x}-3}}{(\cancel{\sqrt{x}-3})(\sqrt{x}+3)}$$

$$= \lim_{x\to 9} \frac{1}{(\sqrt{x}+3)}$$

$$= \frac{\lim_{x\to 9} 1}{\lim_{x\to 9}(\sqrt{x}+3)} \quad \text{by limit rule}$$

$$= \frac{1}{\sqrt{9}+3}$$

$$= \boxed{\frac{1}{6}}$$

(1b) *3 points.* Let $f(x) = \sqrt{x}$, with its natural domain. Does $f'(9)$ exist? Justify very briefly. *You may use problem (1a)!*

$$f'(9) = \lim_{x\to 9} \frac{f(x)-f(9)}{x-9}$$

$$= \lim_{x\to 9} \frac{\sqrt{x}-\sqrt{9}}{x-9}$$

$$= \lim_{x\to 9} \frac{\sqrt{x}-3}{x-9} = \frac{1}{6} \quad \text{(according to (1a))}$$

Thus, $f'(9)$ exists.

(2a) *6 points.* Let $f(x) = \frac{(x-1)(x-4)(x-6)}{5(x-2)(x-3)(x-4)}$, with its natural domain. Find all asymptotes of the graph of f. (You need not sketch the graph, just indicate the types and locations of the asymptotes.)

vertical asymptotes : $x=2$, $x=3$, $\cancel{x=4}$

horizontal asymptotes : $y = \frac{1}{5}$

(2b) *6 points.* Find $\lim_{x\to 0} x\cos(e^{2/x})$. Justify your answer briefly, using any limit rules or theorems from this course.

5

$f(x) = x \Rightarrow$ continuous function

$g(x) = \cos(e^{\frac{2}{x}}) \Rightarrow$ continuous function

Thus, $f \cdot g = x\cos(e^{\frac{2}{x}})$ is continuous

Using Squeeze Theorem

$$-|x| \leq x\cos(e^{\frac{2}{x}}) \leq |x|^2.$$

$$\lim_{x\to 0} x = \lim_{x\to 0} x\cos(e^{\frac{2}{x}}) = \lim_{x\to 0} x^2$$

$$0 = \lim_{x\to 0} x\cos(e^{\frac{2}{x}}) = 0$$

Thus, $\lim_{x\to 0} x\cos(e^{\frac{2}{x}}) = 0$.

6

(3) *6 points.* Show that there is at least one real number x which satisfies $x^6 = 1 + \sin(x)$.

6

continuous

$x^6 = 1 + \sin(x)$ continuous

Let $f(x) = 1 + \sin(x) - x^6$ $\Rightarrow$ continuous

$f(\frac{\pi}{2}) = 1 + 1 - (\frac{\pi}{2})^6 = \ominus$ number

$f(0) = 1 + 0 - 0 = \oplus$ number

Using IMT, there must be a number x

that is $0 < x < \frac{\pi}{2}$ and satisfies $x^6 = 1 + \sin(x)$

since $\underset{\ominus \text{ number}}{f(\frac{\pi}{2})} < \underset{0}{f(x)} < \underset{\oplus \text{ number}}{f(0)}$.

Short answer questions. Only very brief answers are required for these questions. You need not show your work or reasoning. *Each part is worth 2 points.*

(4a) Let $t > 0$. How is $\log_t(3)$ defined?

2

$$\log_t(3) = \boxed{\frac{\ln 3}{\ln t}} \quad (t > 0)$$

$$y = \log_t(3)$$

$$t^y = 3$$

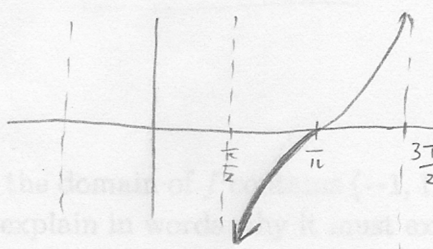
$$y \ln t = \ln 3$$

$$y = \frac{\ln 3}{\ln t}$$

(4b) Let $f(x) = \tan(x)$ with domain $(\frac{\pi}{2}, \pi)$. Does f have an inverse? If so, what are the domain and range of the inverse function?

$$\tan x = \frac{\sin x}{\cos x}$$

2



f has an inverse due to horizontal line test

$D = [0, -\infty)$

$R = (\frac{\pi}{2}, \pi]$

(4c) If some *vertical* line intersects a graph at more than one point, what does this say about the graph?

The graph is not a function.

2

(4d) Simplify: $\ln(5e\sqrt{x})$, assuming that $x > 0$.

$\ln(5e\sqrt{x})$

$= \ln 5 + \ln e\sqrt{x}$

$= \boxed{\ln 5 + \ln\sqrt{x}}$

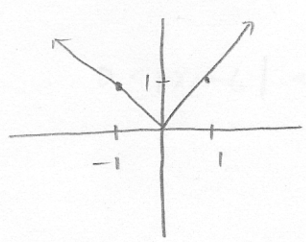
1

(4e) If the domain of f contains $(-1, 1)$, and if f is continuous at 0, must $f'(0)$ exist? Either explain in words why it must exist, or give an example of a function for which it does not exist.

Does not have to exist.

Ex $\quad f(x) = |x|$

2



$f(x) = |x|$ is continuous at 0 and has domain $(-1, 1)$ but $f'(0)$ DNE since it is a corner and has no tangent line.

(5a) *5 points.* Let $f(x) = x^2$. Find $\delta > 0$ such that $|f(x) - 36| < \frac{1}{1000}$ whenever $|x - 6| < \delta$, and **show your reasoning** in full detail. *You need not simplify any numbers which arise, and there is no penalty if your δ is smaller than necessary.*

$$\lim_{x \to 6} x^2 = 36$$

If $0 < |x-6| < \delta$, then $|x^2 - 36| < \varepsilon = \frac{1}{1000}$

$$|x^2 - 36| = |x-6||x+6| < \frac{1}{1000}$$

Let $\delta = 1$, $|x-6| < 1$

$\Rightarrow -1 < x-6 < 1$

$\Rightarrow 5 < x < 7 \Rightarrow 11 < x+6 < 13$

$\Rightarrow |x+6| < 13$

$$|x^2 - 36| = |x+6||x-6| < 13\delta = \varepsilon = \frac{1}{1000}$$

$$\boxed{\delta = \frac{1}{13000}}$$

Proof:

If $0 < |x-6| < \delta$, then $|x^2 - 36| < \varepsilon = \frac{1}{1000}$

$$0 < |x-6| < \frac{1}{13000} \Rightarrow |x-6||x+6| < \frac{1}{13000}(13) = \frac{1}{1000} = \varepsilon$$

(5b) *7 points.* Show, using the precise definition of a limit, that

$$\lim_{x \to \frac{1}{2}} (x - \frac{1}{4x}) = 0.$$

(Here "show" means "prove".)

If $0 < |x - \frac{1}{2}| < \delta$, then $|x - \frac{1}{4x} - 0| < \varepsilon$ (Let $\varepsilon > 0$)

$$|x - \frac{1}{4x}| = \left|\frac{4x^2 - 1}{4x}\right| = \frac{|2x+1||2x-1| \times \frac{1}{2}}{|4x| \times \frac{1}{2}}$$

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$$= \frac{|2x+1|}{|2x|} |x - \tfrac{1}{2}|$$

$$= |1 + \tfrac{1}{2x}||x - \tfrac{1}{2}| < 3\delta = \varepsilon$$

$$\Rightarrow \delta = \tfrac{1}{3}\varepsilon$$

Let $\delta \leq \frac{1}{4}$, then $|x - \frac{1}{2}| < \frac{1}{4} \Rightarrow -\frac{1}{4} < x - \frac{1}{2} < \frac{1}{4}$

$$\Rightarrow \tfrac{1}{4} < x < \tfrac{3}{4}$$

$$\Rightarrow \tfrac{1}{2} < 2x < \tfrac{3}{2}$$

$$\Rightarrow \tfrac{2}{3} < \tfrac{1}{2x} < 2$$

$$\Rightarrow \tfrac{5}{3} < \tfrac{1}{2x} + 1 < 3$$

$$\Rightarrow |\tfrac{1}{2x} + 1| < 3$$

$$\boxed{\delta = \min(\tfrac{1}{4}, \tfrac{1}{3}\varepsilon)}$$

Proof: If $0 < |x - \frac{1}{2}| < \delta$, then $|x - \frac{1}{4x}| < \varepsilon$

$$|x - \tfrac{1}{4x}| = |1 + \tfrac{1}{2x}||x - \tfrac{1}{2}| < 3\delta = \varepsilon$$