Mathematics 1A, Fall 2009 — M. Christ — Final Examination Solutions

There were two versions of the exam. They were very similar, so solutions are given here for only one version.

(1) Calculate the following. **(1a)** The equation of the line tangent to $f(x) = x + e^x$ at $x = 2$. **Solution.** $y = (2 + e^2) + e^2(x - 2)$. □

(1b) $\frac{d}{dx}\sqrt{3 + \ln(\ln(x))}$. **Solution.** $\frac{1}{2}(3 + \ln(\ln(x)))^{-1/2} \cdot \frac{1}{\ln(x)} \cdot \frac{1}{x}$. □

(1c) $\lim_{x\to\pi/2} \frac{\cos(x)}{x-\pi/2}$ **Solution.** $= \lim_{x\to\pi/2} \frac{-\sin(x)}{1} = -\sin(\pi/2) = -1$ by L'Hôpital's rule. □

(1d) $\frac{d}{dx}x^{\cos(x)}$. (Here $x > 0$.)

Solution. $= \frac{d}{dx}e^{\cos(x)\ln(x)} = (-\sin(x)\ln(x) + \cos(x)x^{-1})e^{\cos(x)\ln(x)} = (-\sin(x)\ln(x) + \cos(x)x^{-1})x^{\cos(x)}$. □

(1e) $\int \frac{d}{dx}\sqrt{|\sin(x) + \cos(x)|}\, dx$. **Solution.** $\sqrt{|\sin(x) + \cos(x)|} + C$ where C is an arbitrary constant. □

(1f) $\frac{d}{dx}\int_0^{\sin(2x)} \arcsin(t)\, dt$. **Solution.** $= \arcsin(\sin(2x)) \cdot 2\cos(2x)$.

If $2x$ is in the range $[-\pi/2, \pi/2]$, then this can be simplified, since then $\arcsin(\sin(2x)) = 2x$. If $2x$ is not in this range then $\arcsin(\sin(2x))$ is the unique number t in this range which satisfies $\sin(t) = \sin(2x)$. □

(1h) $\int \sin(x)\cos(x)\, dx$ **Solution.** Substitute $u = \sin(x)$. Then $du = \cos(u)\, du$, and the integral is $\int u\, du = \frac{1}{2}u^2 + C = \frac{1}{2}\sin^2(x) + C$, where C is an arbitrary constant. □

(2j) $\int_0^1 (1 - x^2)^{-1/2}\, dx$ **Solution.** An antiderivative for $(1 - x^2)^{-1/2}$ is $\arcsin(x)$, so by the FTC part II, the integral equals $\arcsin(1) - \arcsin(0) = \frac{\pi}{2} - 0 = \frac{\pi}{2}$. □

(2k) $\sum_{i=1}^4 i^2\cos(\pi i)$ **Solution.** $= 1^2\cos(\pi) + 2^2\cos(2\pi) + 3^2\cos(3\pi) + 4^2\cos(4\pi) = -1 + 4 - 9 + 16 = 10$. □

(2l) $\int_{-2}^2 x\ln(1+x^4)\, dx$ **Solution.** We cannot easily evaluate the indefinite integral $\int x\ln(1+x^4)\, dx$ using techniques from this course. However, the integrand $x\ln(1 + x^4)$ is an odd function, and the integral is from $-a$ to a where $a = 2$, so the definite integral equals 0. □

(2m) $\int (1 - x^2)^{-3/2}\, dx$ (You need not simplify your answer.) **Solution.** Substitute $x = \sin(\theta)$ where $\theta \in (-\pi/2, \pi/2)$. Then $dx = \cos(\theta)\, d\theta$, and $\cos(\theta) = \sqrt{\cos^2(\theta)}$ since $\cos(\theta) \geq 0$. Therefore the integral becomes $\int (\cos(\theta))^{-3}\cos(\theta)\, d\theta = \int \frac{1}{\cos^2(\theta)}\, d\theta = \int \sec^2(\theta)\, d\theta = \tan(\theta) + C = \tan(\arcsin(x)) + C$ where C is an arbitrary constant.

This can be simplified since $\frac{\sin(\arcsin(x))}{\cos(\arcsin(x))} = \frac{x}{\sqrt{1-x^2}}$, but full credit was given for the above answer. □

(2n) $\lim_{x\to\infty}\left((x + x^{1/3})^{2/3} - x^{2/3}\right)$ **Solution.** This is most easily done using L'Hôpital's rule.

$$= \lim_{x\to\infty} x^{2/3}\left((1 + x^{-2/3})^{2/3} - 1\right) = \lim_{x\to\infty} \frac{(1 + x^{-2/3})^{2/3} - 1}{x^{-2/3}} = \lim_{t\to 0^+} \frac{(1 + t^{2/3})^{2/3} - 1}{t^{2/3}}$$

$$= \lim_{t\to 0^+} \frac{\frac{2}{3}(1 + t^{2/3})^{-1/3} \cdot \frac{2}{3}t^{-1/3}}{\frac{2}{3}t^{-1/3}} = \lim_{t\to 0^+} \frac{2}{3}(1 + t^{2/3})^{-1/3} = \frac{2}{3} \cdot (1 + 0)^{-1/3} = \frac{2}{3}. \quad \square$$

(2o) Express an approximation to $\int_1^3 e^{x^2}\,dx$ as a right endpoint Riemann sum with $n = 3$. Your answer need not be simplified; it could be expressed as a sum of several numbers. **Solution.** The endpoints are $a = 1$ and $b = 3$, so $\frac{b-a}{n} = \frac{2}{3}$. Thus $x_1 = 1 + \frac{2}{3} = \frac{5}{3}$, $x_2 = x_1 + \frac{2}{3} = \frac{7}{3}$, $x_3 = 3$. The Riemann sum is

$$\frac{b-a}{n}\sum_{i=1}^{3} e^{x_i^2} = \frac{2}{3}\big(e^{25/9} + e^{49/9} + e^9\big).$$

□

(2p) $\lim_{n\to\infty} \sum_{i=1}^n \frac{n}{n^2+i^2}$. (Either use a method taught in this course, or justify your steps in full detail.) **Solution.**

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{n}{n^2+i^2} = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{n}{n^2+i^2} = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{n^2}{n^2+i^2} = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{1+(i/n)^2}.$$

Let $a = 0$, $b = 1$, and thus $x_i = a + i\frac{b-a}{n} = i/n$. Thus we have $= \lim_{n\to\infty}\frac{b-a}{n}\sum_{i=1}^n \frac{1}{1+x_i^2}$. This is a limit of Riemann sums for

$$\int_0^1 (1+x^2)^{-1}\,dx = \arctan(1) - \arctan(0) = \frac{\pi}{4} - 0 = \frac{\pi}{4}.$$

□

(2q) Use Newton's method with initial approximation $x_1 = 10$ and one step to approximate the cube root of 996. **Solution.** To solve $x^3 = 996$ is the same as solving $f(x) = 0$ where $f(x) = x^3 - 996$.

Taking $x_1 = 10$, the next approximation given by Newton's method is

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 10 - \frac{10^3 - 996}{3\cdot 10^2} = 10 - \frac{4}{300} = 10 - \frac{1}{75}.$$

(Notice that this is slightly less than our initial guess, which makes sense since $10^3 = 1000$ and 996 is slightly less than 1000.) □

Comment. The most common error was to apply Newton's formula using the wrong equation for f. The common mistakes were $f(x) = x^{1/3}$, and $f(x) = x^3$. Newton's formula is used to approximate a solution to the equation $\boxed{f(x) = 0}$. In this problem we were solving $x^3 = 996$, that is, $x^3 - 996 = 0$. So $f(x) = x^3 - 996$. See other examples in text and lecture. □

(3) A right circular cone has height h and has a circular base of radius r. Its volume is $\frac{1}{3}\pi r^2 h$. Suppose that $r^2 + h^2 = 1$. For what value of h is the volume of the cone maximized? What is the maximum volume?

Solution. Express V as a function of h alone: $V(h) = \frac{\pi}{3}h(1-h^2)$. Then $\frac{dV}{dh} = \frac{\pi}{3}(1-h^2) - h\cdot 2h) = \frac{\pi}{3}(1-3h^2)$. This equals zero if and only if $h = 1/\sqrt{3}$. Note that $1/\sqrt{3}$ is in the allowed range $[0,1]$ of values of h.

We also need to check the endpoints: $V(0) = 0$ and $V(1) = 0$. $V(1/\sqrt{3}) = \frac{\pi}{3} \cdot \frac{1}{\sqrt{3}} \cdot (1 - \frac{1}{3}) = \frac{2\pi}{9\sqrt{3}}$. This is greater than the value of V at the endpoints, so the maximum value is $\frac{2\pi}{9\sqrt{3}}$, and it is attained when (and only when) $h = \frac{1}{\sqrt{3}}$. □

Comment. A very straightforward maximization problem. My intent was to give away easy points.

(4) Sketch a graph of the function $f(x) = 4 + xe^{-1/2x}$. Indicate all horizontal and vertical asymptotes (but you need not indicate slant asymptotes), intervals on which f is increasing and decreasing, local maxima and minima, inflection points, and intervals on which the graph is concave up or down. It is not possible to calculate intercepts exactly. Instead, determine exactly how many intercepts there are, and indicate roughly where they are located. You may use the formulas $f'(x) = (1 + \frac{1}{2}x^{-1})e^{-1/2x}$ and $f''(x) = \frac{1}{4}x^{-3}e^{-1/2x}$.

Solution. (Sketch not included here.) (Remember that $e^{-1/2x}$ means $e^{-1/(2x)}$.)

The domain of f is the set of all nonzero real numbers. There is a vertical asymptote at $x = 0$. $f(x) \to 4$ as $x \to 0$ from above, and $f(x) \to -\infty$ as $x \to 0$ from below. $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to \infty$ as $x \to \infty$.

$f'(x) > 0$ for all $x > 0$, and for all $x < -\frac{1}{2}$. $f'(x) < 0$ for all x in $(-\frac{1}{2}, 0)$. So f is increasing on $(0, \infty)$ and on $(-\infty, -\frac{1}{2})$, and decreasing on $(-\frac{1}{2}, 0)$. The only critical point is at $x = -\frac{1}{2}$, and this is a local maximum. $f(-\frac{1}{2}) = 4 - \frac{1}{2}e^{-1/-1} = 4 - \frac{1}{2}e > 0$.

$f(x) > 0$ for all $x > 0$. By the intermediate value theorem, $f(x) = 0$ for at least one x in $(-\infty, -\frac{1}{2})$, and $f(x) = 0$ for at least one x in $(-\frac{1}{2}, 0)$. Because f is increasing on $(-\infty, -\frac{1}{2})$ and decreasing on $(-\frac{1}{2}, 0)$, there can be only one zero in each of those two intervals. So there are exactly two x axis intercepts.

The graph is concave up for $x > 0$ and concave down for $x < 0$; there are no inflection points. □

(5) Newton's law of cooling says: The rate of cooling of a body is proportional to the difference between that body's temperature, and the temperature of its environment. In a cafe where the ambient room temperature is a steady 70 degrees, a cup of coffee is served at 190 degrees. (All temperatures are measured in degrees Fahrenheit.) Assume that Newton's law of cooling applies.

(5a) Let $f(t)$ be the temperature of the coffee at time t. Write a differential equation satisfied by $f(t)$. Your equation may include one or more unknown constants.

Solution. $\frac{df}{dt} = -k(f(t) - 70)$ for some constant k. ($k > 0$ in this problem.) □

Comment. A common error was failure to write any differential equation at all. □

(5b) Write the general solution of your differential equation.

Solution. $f(t) = 70 + Ce^{-kt}$ where C is an arbitrary constant. (It was possible to compute C from the information given, but this was not required; full credit was given for either answer.) □

(5c) 3 minutes after the coffee is served, its temperature is 180 degrees. At what time will the coffee cool to 160 degrees? Express your answer in minutes after the coffee is served.

Solution. Since $f(0) = 190$, $C = 120$. Since $f(3) = 180$, $70 + 120e^{-k\cdot 3} = 180$. Therefore $e^{-3k} = \frac{180-70}{120} = \frac{11}{12}$. Therefore $3k = \ln(12/11)$, so $k = \frac{1}{2}\ln(12/11)$.

Now we solve $160 = f(t) = 70+120e^{-kt}$ with this value of k. We find that $e^{-kt} = \frac{90}{120} = \frac{3}{4}$, so $kt = \ln(4/3)$, so

$$t = \frac{\ln(4/3)}{k} = \frac{\ln(4/3)}{\frac{1}{2}\ln(12/11)} = \frac{2\ln(4/3)}{\ln(12/11)}.$$

□

(6) Show your steps in an organized, legible manner to receive credit. Let $\mathcal{C}$ be the circle with radius 1 centered at $(2, 0)$ in the xy plane. The region enclosed by $\mathcal{C}$ is rotated around the y axis to generate a three dimensional solid known to mathematicians as a torus, and to law enforcement officers as a staple food.

(6a) Using the method of cylindrical shells, express the volume of this solid as an integral.

Solution. $V = 2\pi \int_1^3 x\cdot 2\sqrt{1-(x-2)^2}\,dx$. (Some students missed a factor of 2; they didn't take into account the part of the solid in the region $y < 0$. One point was deducted.) □

(6b) Evaluate this integral, using methods and results taught in this course. (You may not be able to find an antiderivative, but it is possible to evaluate the definite integral using material taught in this course.)

Solution. Substitute $x = t + 2$ to write the integral as

$$4\pi \int_{-1}^{1} (t+2)\sqrt{1-t^2}\,dt = 4\pi \int_{-1}^{1} t\sqrt{1-t^2}\,dt + 4\pi \int_{-1}^{1} 2\sqrt{1-t^2}\,dt.$$

Now $\int_{-1}^{1} 2\sqrt{1-t^2}\,dt$ represents the area of the circle (or disk) $x^2 + y^2 \le 1$ in the xy plane; this area is π, as we calculated in lecture one day in late November. The integral $\int_{-1}^{1} t\sqrt{1-t^2}\,dt$ is the integral of an odd function over an interval of the form $[-1, 1]$, so it equals zero! Therefore in total we find that the volume equals

$$4\pi \int_{1}^{3} x\cdot 2\sqrt{1-(x-2)^2}\,dx = 4\pi \int_{-1}^{1} t\sqrt{1-t^2}\,dt + 4\pi \int_{-1}^{1} 2\sqrt{1-t^2}\,dt = 4\pi\cdot 0 + 4\pi\cdot\pi = 4\pi^2.$$

□

Comment. There is an alternative approach: It is possible to evaluate $\int t\sqrt{1-t^2}\,dt$ via the substitution $u = 1 - t^2$. We get $du = -2t\,dt$, so $t\,dt = -\frac{1}{2}\,du$, and the integral becomes

$$-\tfrac{1}{2}\int u^{1/2}\,du = -\frac{1}{2}\cdot\frac{2}{3}u^{3/2} + C = -\tfrac{1}{3}(1-t^2)^{3/2} + C$$

where C is an arbitrary constant. This function has the same value at $+1$ as at -1, so by plugging in the endpoints of integration and subtracting, we get 0.

This is justified by the formula $\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$. □

Comment. Credit was given in part (b) for progress on any integral which was close to the correct answer to part (a), or was equally challenging, but not for work on incorrect answers to part (a) which were easier to evaluate than was the correct integral. □

(7a) An emu and a wombat race along a straight line, beginning at time $t = 0$. The wombat is given a head start. At time t, their positions are $E(t)$ and $W(t)$, respectively. Suppose that $W(t) = 1 + t$ for all $t \geq 0$, that $E(0) = 0$, and that $E''(t) < 0$ for all $t \geq 0$. What is the maximum possible number of times $t > 0$ at which $E(t)$ can be equal to $W(t)$? Explain your answer briefly.

Answer. The function $f(t) = E(t) - W(t)$ can vanish for at most two times t, because $f''(t) = E''(t) - 0 < 0$ for all t. If f were to vanish at $t_1 < t_2 < t_3$, then by Rolle's theorem, f' would vanish at some time c_1 in (t_1, t_2), and also at some time c_2 in (t_2, t_3). Applying Rolle's Theorem to f', we would conclude that $f''(c) = 0$ for some c in (c_1, c_2). This would contradict the fact that f'' is always negative. □

(7b) Define: f has an inflection point at x.

Answer. f has an inflection point at x if the direction of concavity of the graph of f changes at x, either from convex up to convex down, or vice versa. □

(7c) Let t = time, $s(t)$ = the position of a projectile at time t, and $v(t)$ = its velocity. Let $a < b$ be two times. We have defined two kinds of averages of v over the interval $[a, b]$: (i) The net change in position divided by the elapsed time, and (ii) $(b - a)^{-1} \int_a^b v(t)\, dt$. How are these two averages related to one another? Explain very briefly.

Answer. They are equal. By the FTC,

$$\int_a^b v(t)\, dt = s(b) - s(a) = \text{ the net change in position.}$$

Since $b - a$ = the elapsed time, dividing both sides by $b - a$ shows that the two averages are equal. □

(7d) If $f(0) = 0$, $f'(0) = -1$, and $f''(x) \leq 2$ for all x, what is the largest possible value of $f(3)$?

Answer. $f(3) \leq 6$. Consider the function $g(x) = f(x) - (-x + x^2)$. Then $g(0) = g'(0) = 0$, and $g''(x) \leq f''(x) - 2 \leq 0$ for all x. Therefore $g'(x) \leq g'(0) = 0$ for all $x \geq 0$. Therefore g is a decreasing function on $[0, \infty)$. Therefore $g(3) \leq g(0) = 0$. But $g(3) = f(3) - (-3 + 9) = f(3) - 6$. So $f(3) - 6 \leq 0$. On the other hand, the function $f(x) = -x + x^2$ satisfies all of the assumptions, and satisfies $f(3) = 6$. □

(7e) If $f'(-1) = f'(1)$, and if $f''(x)$ exists and is a continuous function on $[-1, 1]$, then two conclusions can be drawn about f''. What are they?

Answer. First, $f''(c) = 0$ for some c in $(-1, 1)$ (Rolle's Theorem). Second, $\int_{-1}^1 f''(x)\, dx = f'(1) - f'(-1) = 0$. □

(7f) If f and its derivative f' are continuous functions defined for all real numbers, and if $f'(x + 1) = f'(x)$ for all x, what conclusion can be drawn about f?

Answer. $f(x + 1) - f(x)$ is a constant, independent of x. Indeed,

$$f(x+1) - f(x) = \int_0^{x+1} f'(t)\, dt - \int_0^x f'(t)\, dt = \int_0^1 f'(t)\, dt + \int_1^{x+1} f'(t)\, dt - \int_0^x f'(t)\, dt.$$

But by substituting $t = s + 1$ gives

$$\int_1^{x+1} f'(t)\,dt = \int_0^x f'(s+1)\,sx = \int_0^x f'(s)\,ds,$$

using the assumption about f'. So the last two integrals cancel, leaving $f(x+1) - f(x) = \int_0^1 f'(t)\,dt$. (Detailed justification was not required for credit. We had discussed an example of this type in class as part of the unit on graph sketching.) □

(7g) Let f be a continuous function defined for all $x \geq 0$. For $s > 0$ let $g(s) = s^{-1}\int_0^s f(x)\,dx$. Suppose that t is a positive number such that $g(t) \geq g(s)$ for every $s > 0$. Find an equation relating $f(t)$ to $g(t)$.

Answer. $f(t) = g(t)$. Indeed, the assumption says that g has a local maximum at t. Therefore $g'(t) = 0$. By the FTC and the product rule,

$$g'(t) = -t^{-2}\int_0^t f(x)\,dx + t^{-1}f(t).$$

Since $g'(t) = 0$, multiplying through by t gives

$$0 = -t^{-1}\int_0^t f(x)\,dx + f(t) = -g(t) + f(t).$$

□

(7h) Explain briefly how the formula $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ in Newton's method is derived.

Answer. In Newton's method, one wants to solve $f(x) = 0$, approximately. The linear approximation says that $f(x_{n+1})$ is approximately equal to $f(x) + (x - x_n)f'(x_n)$. Setting this approximation (rather than f itself) equal to zero and solving for x leads to Newton's formula. □

(7i) If you were asked, as the final problem on this exam, to derive the formula $\operatorname{arcsec}'(x) = \frac{1}{x\sqrt{x^2-1}}$, assuming that arcsec is differentiable, how would you begin? You need not give a proof or do any calculations, but show that you know what to calculate.

Answer. $\sec(\operatorname{arcsec}(x)) = x$ for all x in the domain of arcsec. Apply $\frac{d}{dx}$ to both sides of this equation, and use the chain rule (*full credit for getting this far!*) to get

$$\sec'(\operatorname{arcsec}(x)) \cdot \operatorname{arcsec}'(x) = 1.$$

Solve for $\operatorname{arcsec}'(x)$, use $\sec'(x) = \tan(x)$, and simplify $\tan(\operatorname{arcsec}(x))$. □