ME132 Spring 2007 Midterm 1 Solutions

$\boxed{\text{Problem 1}}$

$\dot{x} = -3x + 6u$

steady state gain : $\frac{x_{ss}}{u_{ss}} = +2$

time constant : $\tau = 1/3 \longrightarrow$ ~95% reduction in error after $3\tau = 1$ sec

Problem 2

$\dot{y}(t) + ay(t) = bu(t) \qquad a>0,\ b>0$

1) Let $u(t) = u_R(t) + j u_I(t)$
$y_p(t) = y_{PR}(t) + j y_{PI}(t)$

Then, since $y_p(t)$ satisfies the ODE:

$$\frac{d}{dt}(y_{PR} + j y_{PI}) + a(y_{PR} + j y_{PI}) = b(u_R + j u_I)$$

$$\dot{y}_{PR} + j\dot{y}_{PI} + a(y_{PR} + j y_{PI}) = b(u_R + j u_I)$$

Taking the real and imaginary parts of this equation yields:

$$\boxed{\begin{aligned} \dot{y}_{PR} + a y_{PR} &= b u_R \\ \dot{y}_{PI} + a y_{PI} &= b u_I \end{aligned}}$$

2) $u(t) = e^{j\omega t}$

a) Let $y_p(t) = G(\omega)u(t) = G(\omega)e^{j\omega t}$

Plugging $y_p(t)$ into the original ODE gives:

$$\frac{d}{dt}(G(\omega)e^{j\omega t}) + aG(\omega)e^{j\omega t} = be^{j\omega t}$$

$$j\omega G(\omega)e^{j\omega t} + aG(\omega)e^{j\omega t} = be^{j\omega t}$$

$$[G(\omega)[j\omega + a]]e^{j\omega t} = be^{j\omega t}$$

$$\boxed{G(\omega) = \frac{b}{j\omega + a}}$$

b) From (a), $y_p(t) = \frac{b}{j\omega + a} e^{j\omega t}$

which can be written as $y_p(t) = M(\omega)e^{j(\omega t + \phi(\omega))}$

so $M(\omega) = |G(\omega)|$ and $\phi(\omega) = \angle G(\omega)$

$M(\omega) = \left|\frac{b}{j\omega + a}\right| = \frac{b}{|j\omega + a|}$ since $b>0$ from the problem statement

$$\boxed{M(\omega) = \frac{b}{(\omega^2 + a^2)^{1/2}}}$$

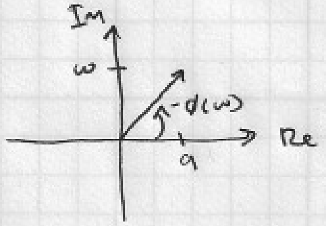
Problem 2 continued

and $\phi(\omega) = \angle G(\omega) = \angle \frac{b}{j\omega + a}$

$= \angle b - \angle (j\omega + a)$

$= 0 - \tan^{-1}\left(\frac{\omega}{a}\right)$

$$\boxed{\phi(\omega) = -\tan^{-1}\left(\frac{\omega}{a}\right)}$$



3) $u(t) = \cos \omega t = Re\left[e^{j\omega t}\right]$

a) We already found $y_p(t)$ for $u(t) = e^{j\omega t}$, so if we want to find $y_p(t)$ for $u(t) = Re[e^{j\omega t}]$, all we have to do is take the real part of the $y_p(t)$ we found earlier, i.e.

$Re\left[\dot{y}_p(t) + a\, y_p(t) = b\, u(t)\right]$

or

$y_p(t) = Re\left[M(\omega)\, e^{j(\omega t + \phi(\omega))}\right]$

$= Re\left[M(\omega)\left[\cos(\omega t + \phi(\omega)) + j \sin(\omega t + \phi(\omega))\right]\right]$

$= M(\omega) \cos(\omega t + \phi(\omega))$

So $\boxed{M(\omega) = \frac{b}{(\omega^2 + a^2)^{1/2}}, \quad \phi(\omega) = -\tan^{-1}\left(\frac{\omega}{a}\right)}$ as before

b) · ω can be read from the input $u(t)$ and the formula relating period T to angular frequency ω.

$T = 1 = \frac{2\pi}{\omega} \Rightarrow \boxed{\omega = 2\pi}$

· $\phi(\omega)$ can be found by looking at the phase difference and then a can be found by using the formula for $\phi(\omega)$.

$\left.\begin{array}{l} t_{lag} = -1/8 \\ T = 1 \end{array}\right\} \phi(\omega) = 2\pi\left(\frac{-1/8}{1}\right) = -\pi/4$

$-\pi/4 = -\tan^{-1}\left(\frac{\omega}{a}\right)$

$a = \omega$

$\boxed{a = 2\pi}$

- b can be found by looking at the steady state gain from $u \to y$ and setting it equal to $M(\omega)$.

$$M(\omega) = 1$$

$$\frac{b}{\left((2\pi)^2 + (2\pi)^2\right)^{1/2}} = 1$$

$$b = \left(4\pi^2 + 4\pi^2\right)^{1/2}$$

$$\boxed{b = \sqrt{8}\,\pi = 2\sqrt{2}\,\pi}$$

Problem 3

$$\ddot{\theta}(t) + \alpha\dot{\theta}(t) = E u(t) + G d(t)$$

1) $\theta(0) \neq \bar{\theta}_{des}$

In general, it is not possible to acheive the control objective because the system is, at best, limitedly stable so open loop control will not force $\theta \to \bar{\theta}_{des}$ for some set of initial conditions. However, there is a special case where the control objective can be acheived with O.L. control. Note that the solution $y(t)$ looks like:

$$y(t) = y_H(t) + y_P(t)$$
$$= c_1 e^{-\alpha t} + c_2 + \frac{1}{\alpha}(E\bar{u}_{op} + G\bar{d})t$$

Letting $\theta(0) = \theta_0$, $\dot{\theta}(0) = \dot{\theta}_0$, and noting that $\bar{u}_{op} = -\frac{G}{E}\bar{d}$ is the only $\bar{u}_{op}$ that allows $\theta \to \theta_{ss}$ (not unstable)

$$y(t) = -\frac{\dot{\theta}_0}{\alpha}e^{-\alpha t} + \left(\theta_0 + \frac{\dot{\theta}_0}{\alpha}\right)$$

As $t \to \infty$, $e^{-\alpha t} \to 0$, so

$$\theta_{ss} = \theta_0 + \frac{\dot{\theta}_0}{\alpha}$$

If we want $\theta_{ss} = \bar{\theta}_{des}$, then we need

$$\dot{\theta}_0 = \alpha(\theta_{ss} - \theta_0)$$

Thus, it is theoretically possible to have $\theta_{ss} \to \bar{\theta}_{des}$ (with a correct choice of I.C.s) but in general, the control objective is not possible.

Problem 3 continued

2) $u(t) = K_P[\theta_{des} - \theta(t)] - K_D\dot{\theta}(t)$

a) Plug u(t) into the θ(t) ODE:

$$\ddot{\theta} + \alpha\dot{\theta} = EK_P[\theta_{des} - \theta] - EK_D\dot{\theta} + Gd$$

$$\boxed{\ddot{\theta}(t) + [\alpha + EK_D]\dot{\theta}(t) + EK_P\theta(t) = EK_P\theta_{des} + Gd(t)}$$

b) Use the 2nd order conditions for stability:

$\alpha + EK_D > 0$ $\qquad$ $EK_P > 0$

$K_D > \frac{-\alpha}{E}$ $\qquad$ $\boxed{K_P > 0}$

-or-

$\boxed{K_D > -0.1}$

c) $d(t) \equiv 0$, $\theta_{des} = \bar{\theta}_{des}$

With K_P and K_D selected so the homogeneous solution is asymptotically stable, then we know that $\theta \rightarrow \bar{\theta} = \text{constant}$.

so $\dot{\bar{\theta}} = \ddot{\bar{\theta}} = 0$

Using these conditions in the ODE from (a), we get

$$\cancelto{0}{\ddot{\bar{\theta}}} + [\alpha + EK_D]\cancelto{0}{\dot{\bar{\theta}}} + EK_P\bar{\theta} = EK_P\bar{\theta}_{des} + 0$$

$$EK_P\bar{\theta} = EK_P\bar{\theta}_{des}$$

$$\boxed{\bar{\theta} = \bar{\theta}_{des}}$$

d) we know that for a 2nd order stable system:

$\alpha + EK_D = 2\xi\omega_n$ $\qquad$ $EK_P = \omega_n^2$

$K_D = \frac{2\xi\omega_n - \alpha}{E}$ $\qquad$ $K_P = \frac{\omega_n^2}{E}$

Plugging in $\xi = .707$, $\omega_n = 10$, and the ODE parameters:

$K_D = \frac{2(.707)10 - 0.1}{1}$ $\qquad$ $K_P = \frac{10^2}{1}$

$\boxed{K_D = 14.04}$ $\qquad$ $\boxed{K_P = 100}$

Problem 3 continued

e) Using the conditions outlined in (c) and (d), we can plot the C.L. response.

$$\tau = \frac{1}{\zeta \omega_n} \approx .141$$

$\rightarrow$ settles in about $3\tau \simeq .424$

Steady state gain $\bar{\theta} = \bar{\theta}_{des}$