MATH 53 — MIDTERM #2

Each problem counts 20 points.

Problem #1. Calculate the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

for the vector field $\mathbf{F} = < -y, x >$ and the half circle C given parametrically by $\mathbf{r}(t) = < \cos t, \sin t >$ for $0 \leq t \leq \pi$.

Problem #2. Find all solutions (x, y, z, λ) of the equations given by the Lagrange multiplier method for the problem of determining the points on the surface $z^2 = xy + 4$ closest to the origin.

Of these solutions, which give the points closest to the origin?

Problem #3. For the vector field $\mathbf{F} = < yz + x^2, xz + y, xy + z >$ compute the value of

$$\int_C \mathbf{F} \cdot d\mathbf{r},$$

where C is any curve connecting the point $(1, 2, 0)$ to $(2, 0, 3)$.

Problem #4. Use the transformation

$$x = 2u, \quad y = 4u + v$$

to evaluate the integral

$$\iint_R (2x(y - 2x))^{\frac{1}{2}} \, dA,$$

where R is the parallelogram with vertices $(0, 0), (0, 1), (2, 4), (2, 5)$.

Problem #5. Find the volume of the solid lying within the sphere $\rho = 4\cos\phi$ and below the cone $\phi = \frac{\pi}{4}$.

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1. $\vec{r}'(t) = \langle -\sin t, \cos t \rangle$ — 0

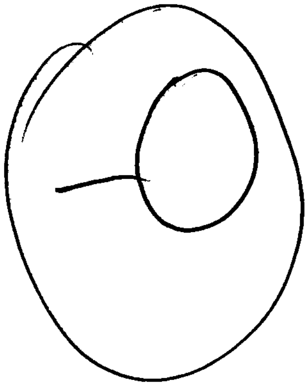
$\vec{F}(\vec{r}(t)) = \langle -\sin t, \cos t \rangle$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{\pi} \langle -\sin t, \cos t \rangle \cdot \langle -\sin t, \cos t \rangle \, dt$$

$$= \int_0^{\pi} \sin^2 t + \cos^2 t \, dt$$

$$= \int_0^{\pi} dt$$

$$= \boxed{\pi}$$



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2. $\begin{cases} d^2 = x^2+y^2+z^2 \\ g(x,y,z) = z^2 - xy = 4 \end{cases} \Rightarrow \begin{cases} 2x = -\lambda y & ① \\ 2y = -\lambda x & ② \\ 2z = 2\lambda z & ③ \\ z^2 - xy = 4 & ④ \end{cases}$

②: $y = -\frac{\lambda}{2}x$ ⑤

⑤→①: $2x = -\lambda(-\frac{\lambda}{2})x$

$2x = \frac{\lambda^2}{2}x$

$(\lambda^2-4)x = 0 \Rightarrow \lambda^2 = 4$ or $x = 0$

if $x=0$, $y = -\frac{\lambda}{2}x = 0$

$z^2 = 4 \Rightarrow z = \pm 2$

$2z = 2\lambda z \Rightarrow \lambda = 1$

$\Rightarrow \begin{cases} x=0 \\ y=0 \\ z=\pm 2 \\ \lambda = 1 \end{cases}$

if $\lambda^2 = 4$, if $\lambda = 2$, $2x = -2y \Rightarrow x = -y$

$2z = 4z \Rightarrow z = 0$

$z^2 - xy = 4 \Rightarrow -(-y)\cdot y = 4 \Rightarrow y^2 = 4 \Rightarrow y = \pm 2$

$\Rightarrow \begin{cases} x = \pm 2 \\ y = \mp 2 \\ z = 0 \\ \lambda = 2 \end{cases}$

if $\lambda = -2$, $2x = 2y \Rightarrow x = y$

$2z = -4z \Rightarrow z = 0$

$z^2 - xy = 4 \Rightarrow -y^2 = 4 \Rightarrow$ no solution

$d(0,0,\pm 2) = \sqrt{0+0+4} = 2$

$d(\pm 2, \mp 2, 0) = \sqrt{4+4+0} = 2\sqrt{2}$

Solutions: $\begin{cases} x=0 \\ y=0 \\ z=\pm 2 \\ \lambda=1 \end{cases}$ or $\begin{cases} x=2 \\ y=-2 \\ z=0 \\ \lambda=2 \end{cases}$ or $\begin{cases} x=-2 \\ y=2 \\ z=0 \\ \lambda=2 \end{cases}$

Closest: $\begin{cases} x=0 \\ y=0 \\ z=\pm 2 \\ \lambda=1 \end{cases}$

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3. $P = yz + x^2$ $\qquad Q = xz + y$ $\qquad R = xy + z$

$\frac{\partial P}{\partial y} = z$ $\qquad \frac{\partial Q}{\partial x} = z$ $\qquad \frac{\partial R}{\partial x} = y$ — 0

$\frac{\partial P}{\partial z} = y$ $\qquad \frac{\partial Q}{\partial z} = x$ $\qquad \frac{\partial R}{\partial y} = x$

$$\left.\begin{array}{l} \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \\ \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} \\ \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y} \end{array}\right\} \Rightarrow \vec{F} \text{ is a conservative vector field}$$

$$\left.\begin{array}{l} \frac{\partial f}{\partial x} = yz + x^2 \Rightarrow f = xyz + \frac{1}{3}x^3 + C_1 \\ \frac{\partial f}{\partial y} = xz + y \Rightarrow f = xyz + \frac{1}{2}y^2 + C_2 \\ \frac{\partial f}{\partial z} = xy + z \Rightarrow f = xyz + \frac{1}{2}z^2 + C_3 \end{array}\right\} \Rightarrow f = xyz + \frac{1}{3}x^3 + \frac{1}{2}y^2 + \frac{1}{2}z^2$$

$$\int_C \vec{F} \cdot d\vec{r} = f(2,0,3) - f(1,2,0)$$
$$= \left(\frac{8}{3} + \frac{9}{2}\right) - \left(\frac{1}{3} + 2\right) = \frac{16+27}{6} - \frac{7}{3} = \frac{43-14}{6} = \boxed{\frac{29}{6}}$$

-0 4

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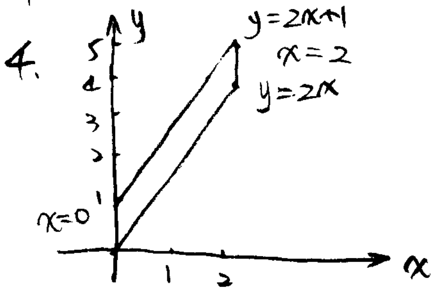
4.



$x=0 \Rightarrow 2u=0 \Rightarrow u=0$

$x=2 \Rightarrow 2u=2 \Rightarrow u=1$

$y=2x \Rightarrow \cancel{4u}+v=\cancel{4u} \Rightarrow v=0$

$y=2x+1 \Rightarrow \cancel{4u}+v=\cancel{4u}+1 \Rightarrow v=1$

$D=\{(u,v) \mid 0 \le u \le 1,\ 0 \le v \le 1\}$

$\frac{\partial x}{\partial u}=2 \quad \frac{\partial x}{\partial v}=0$

$\frac{\partial y}{\partial u}=4 \quad \frac{\partial y}{\partial v}=1$

$\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix} 2 & 0 \\ 4 & 1 \end{vmatrix}=2$

$\iint_R (2x(y-2x))^{\frac{1}{2}}\,dA$

$=\int_0^1\int_0^1 [2\cdot(2u)(\cancel{4u}+v-\cancel{4u})]^{\frac{1}{2}}\cdot 2\,du\,dv$

$=\int_0^1\int_0^1 \sqrt{4uv}\cdot 2\,du\,dv = 4\int_0^1\int_0^1 \sqrt{uv}\,du\,dv$

$=4\int_0^1 \frac{2}{3v}(uv)^{\frac{3}{2}}\Big|_0^1\,dv = \frac{8}{3}\int_0^1 \sqrt{v}\,dv$

$=\frac{8}{3}\times\frac{2}{3}v^{\frac{3}{2}}\Big|_0^1 = \frac{8}{3}\times\frac{2}{3} = \boxed{\frac{16}{9}}$

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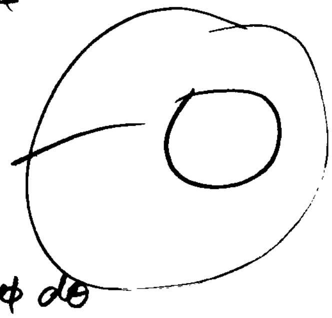

5. $\rho \geq 0 \Rightarrow 4\cos\phi \geq 0 \Rightarrow 0 \leq \phi \leq \frac{\pi}{2}$ below the cone $\Rightarrow \frac{\pi}{4} \leq \phi \leq \pi$

$S = \{(\rho, \theta, \phi) \mid 0 \leq \rho \leq 4\cos\phi,\ \frac{\pi}{4} \leq \phi \leq \frac{\pi}{2}\ \ 0 \leq \theta \leq 2\pi\}$

$$V = \int_0^{2\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{4\cos\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{3}\rho^3 \sin\phi \Big|_0^{4\cos\phi} d\phi \, d\theta = \frac{1}{3}\int_0^{2\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 64\cos^3\phi \sin\phi \, d\phi \, d\theta$$

$$= \frac{64}{3}\int_0^{2\pi} -\frac{1}{4}\cos^4\phi \Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} d\theta = -\frac{16}{3}\int_0^{2\pi}\left(0 - \frac{1}{4}\right) d\theta = \frac{16}{3} \times \frac{1}{4}\int_0^{2\pi} d\theta$$

$$= \frac{4}{3}\int_0^{2\pi} d\theta = \boxed{\frac{8}{3}\pi}$$