Abstract Algebra 113 - Midterm 1

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Question 1

Let G be a set.

1. What is a binary operation on G?
2. Carefully define what it means for a set G with a binary operation $*$ to be a group.
3. Let $(G, *)$ be a group and $g \in G$. Prove that the map

$$\begin{aligned} \phi_g : G &\rightarrow G \\ h &\rightarrow h * g \end{aligned}$$

   is a bijection.

4. For what possible values of g is ϕ_g a homomorphism. Carefully justify your answer.

Question 2

Let $(G, *)$ be a group and S a set.

1. What is an *action* of G on S?
2. Assume we are given an action φ, of G on S. Let $s \in S$. Define $stab(s) \subset G$ and $orb(s) \subset S$.
3. State (without proof) the orbit-stabiliser theorem (G is not necessarily finite).
4. If $|G| = 15$ what are the possible sizes of $|orb(s)|$? Justify your answer.

Question 3

Let $(G, *)$ be a finite group, and H a subgroups. In this question you may use either equivalent definition of a right coset of H in G.

1. Assume that H is of finite index in G. Prove that there exists $n \in \mathbb{N}$ and $\{x_0, \cdots x_n\} \subset G$ with the property that given $g \in G \; \exists \; i \in \{0, \cdots n\}$ and $h \in H$ such that $g = x_i * h$.
2. State, without proof, Lagrange's Theorem.
3. Prove that all finite groups of prime order are cyclic.

Question 4

(In this question you may use any result in the lectures as long as you state it clearly). Let $(G, *)$ be an cyclic group.

1. If G is infinite prove that it has exactly two possible generators.
2. Given $n \in \mathbb{N}$, prove that there is a finite cyclic group which has more than n distinct generators.

Cyclic $\Rightarrow$ for some $x \in G$, $gp(\{x\}) = G$