Midterm Solutions

1. A

$$\begin{array}{ll} \min & f(x) = (x-a)^2 \\ s.t. & x \geq b \\ & x \leq c \end{array}$$

The K-T conditions yield

$$2(x-a) - \lambda_1 - \lambda_2 = 0$$
$$\lambda_1 \geq 0 \quad (1)$$
$$\lambda_2 \leq 0 \quad (2)$$
$$\lambda_1(x-b) = 0$$
$$\lambda_2(x-c) \geq 0$$

Here we can see that x can only be b, c or between b and c.
If $x^* = b, \lambda_2 = 0, \lambda_1 = 2(x^* - a) = 2(b-a) \quad i)$
If $x^* = c, \lambda_1 = 0, \lambda_2 = 2(x^* - a) = 2(c-a) \quad ii)$
If $b < x^* < c, \lambda_1 = \lambda_2 = 0, x^* = a \quad iii)$

Here we begin to discuss in cases.
1) $a < b$
i) holds
ii) $\lambda_2 > 0$, contradictory to (2)
iii) $x^* = a$, contradictory to $b < x^* < c$
So $x^* = b, \lambda_1 = 2(b-a), \lambda_2 = 0, f(x^*) = (b-a)^2$

2) $b < a < c$
i) $\lambda_1 < 0$, contradictory to (1)
ii)$\lambda_2 > 0$, contradictory to (2)
iii)holds
So $x^* = a, \lambda_1 = \lambda_2 = 0, f(x^*) = 0$

3) $c < a$
i) $\lambda_1 < 0$, contradictory to (1)
ii) holds
iii) $x^* = a$, contradictory to $b < x^* < c$
So $x^* = c, \lambda_1 = 0, \lambda_2 = 2(c-a), f(x^*) = (c-a)^2$

B

$$\begin{array}{ll} \max & f(x) = (x-a)^2 \\ s.t. & x \geq b \\ & x \leq c \end{array}$$

The K-T conditions yield

$$\begin{aligned} 2(x-a) - \lambda_1 - \lambda_2 &= 0 \\ \lambda_1 &\leq 0 \qquad (3) \\ \lambda_2 &\geq 0 \qquad (4) \\ \lambda_1(b-x) &= 0 \\ \lambda_2(c-x) &\geq 0 \end{aligned}$$

Similarly x can only be b, c or between b and c.
If $x^* = b, \lambda_2 = 0, \lambda_1 = 2(x^* - a) = 2(b-a) \quad i)$
If $x^* = c, \lambda_1 = 0, \lambda_2 = 2(x^* - a) = 2(c-a) \quad ii)$
If $b < x^* < c, \lambda_1 = \lambda_2 = 0, x^* = a \quad iii)$

Here we begin to discuss in cases.
1) $a < b$
i) $\lambda_1 > 0$, contradictory to (3)
ii) holds
iii) $x^* = a$, contradictory to $b < x^* < c$
So $x^* = c, \lambda_1 = 0, \lambda_2 = 2(c-a), f(x^*) = (c-a)^2$

2) $b < a < c$
i) holds
ii)holds
iii)holds
But since $f(x)$ is convex, $x = a$ is a minimum, then we are left with cases $x^* = b$ or $x^* = c$. Which one is the solution depends on a.
If $b < a < (b+c)/2, x^* = c, \lambda_1 = \lambda_2 = 0, f(x^*) = (c-a)^2$
If $(b+c)/2 < a < c, x^* = b, \lambda_1 = \lambda_2 = 0, f(x^*) = (b-a)^2$

3) $c < a$
i) holds
ii) $\lambda_2 < 0$, contradictory to (4)
iii) $x^* = a$, contradictory to $b < x^* < c$
So $x^* = b, \lambda_1 = 2(b-a), \lambda_2 = 0, f(x^*) = (b-a)^2$

2. $d = \frac{-\bigtriangledown f(x)}{||\bigtriangledown f(x)||}$ for $\bigtriangledown f(x) \neq 0$.

3. Let $s_j = \{x | g_j(x) \leq b_j\}$. Since $g_j(x)$ is convex, $\forall x, y \in s_j, 0 \leq \lambda \leq 1$,

$$g_j(\lambda x + (1-\lambda)y) \leq \lambda g_j(x) + (1-\lambda) g_j(y)$$

Since $g_j(x) \leq b_j$ and $g_j(y) \leq b_j$,

$$g_j(\lambda x + (1-\lambda)y) \leq \lambda b_j + (1-\lambda) b_j = b_j$$

So $\lambda x + (1-\lambda)y \in s_j$ and s_j is convex.
$\forall x, y \in S, 0 \leq \lambda \leq 1$. Since $S = \bigcap_{j=1}^{n} s_j$, it follows that $x, y \in s_j$, for $j = 1, ..., n$. So

$$\lambda x + (1-\lambda)y \in s_j$$

and

$$\lambda x + (1-\lambda)y \in \bigcap_{j=1}^{n} s_j = S$$

thus S is convex.

4. We need to solve the problem

$$\begin{array}{ll} \max & -x_1^2 + 8x_1 - x_2^2 + 12x_2 \\ \text{s.t.} & x_1 + x_2 \le 8 \\ & 0 \le x_1 \le 5 \\ & 0 \le x_2 \le 4 \end{array}$$

First note that the objective function is concave and the feasible set is convex, therefore any solution of the KT conditions must be a global solution. The KT conditions are:

$$\begin{array}{l} -2x_1 + 8 - \lambda_1 - \lambda_2 - \lambda_3 = 0 \\ -2x_2 + 12 - \lambda_1 - \lambda_4 - \lambda_5 = 0 \\ \lambda_1(8 - x_1 - x_2) = 0 \\ -\lambda_2 x_1 = 0 \\ \lambda_3(5 - x_1) = 0 \\ -\lambda_4 x_2 = 0 \\ \lambda_5(4 - x_2) = 0 \\ \lambda_1, \lambda_3, \lambda_5 \ge 0,\ \lambda_2, \lambda_4 \le 0 \text{ plus the original constraints.} \end{array}$$

Alternatively, we can formulate the KT conditions as:

$$\begin{array}{l} -2x_1 + 8 - \lambda_1 - \lambda_3 \le 0 \\ -2x_2 + 12 - \lambda_1 - \lambda_5 \le 0 \\ \lambda_1(8 - x_1 - x_2) = 0 \\ -(-2x_1 + 8 - \lambda_1 - \lambda_3)x_1 = 0 \\ \lambda_3(5 - x_1) = 0 \\ -(-2x_2 + 12 - \lambda_1 - \lambda_5)x_2 = 0 \\ \lambda_5(4 - x_2) = 0 \\ \lambda_1, \lambda_3, \lambda_5 \ge 0, \text{ plus the original constraints.} \end{array}$$

Since the unconstrained optimal solution is attained at (4,6), we can expect that the non-negativity constraints are not biding, therefore we assume that $\lambda_2 = \lambda_4 = 0$. Trying also $\lambda_1 = \lambda_3 = 0$ and $\lambda_5 > 0$ we find that $x_1 = x_2 = 4$ and $\lambda_5 = 4$ satisfy the KT conditions. Then (4,4) is the optimal solution. Since the only positive Lagrange multiplier is λ_5, the only constraint that changes the optimal objective when the RHS is changed is $x_2 \le 4$ and the rate of change is 4. Note that even though the constraint $x_1 + x_2 \le 8$ is tight, if we increase the RHS the objective function does not change.

5. Let z be the point on the y axis where the runner enter the water. Then the time spend from s to f is:

$$g(z) = \frac{\sqrt{x_1^2 + (z - y_1)^2}}{v_1} + \frac{\sqrt{x_2^2 + (z - y_2)^2}}{v_2}.$$

We want to minimize $g(z)$. Note that g is the sum of two convex functions. Since $-x_1, x_2 > 0$, the function g is everywhere differentiable. Thus the optimality condition in this case is:

$$g'(z) = \frac{z - y_1}{v_1\sqrt{x_1^2 + (z - y_1)^2}} + \frac{z - y_2}{v_2\sqrt{x_2^2 + (z - y_2)^2}} = 0.$$

If z^* is such that $g'(z^*) = 0$, then z^* is the global solution.