Mathematics 54.1
Midterm 1, 17 February 2011
75 minutes, 75 points

INSTRUCTIONS:
You must justify your answers, except when told otherwise.
All the work for a question should be on the respective sheet.
This is a CLOSED BOOK examination, NO NOTES and NO CALCULATORS are allowed.
NO CELL PHONE or EARPHONE use is permitted.
Please turn in your finished examination to your GSI before leaving the room.

1. TRUE-FALSE Questions (36 points) $\forall\exists$

Circle the correct answer, no justification needed. Correct answers carry 1.5 points, wrong answers carry 1.5 points penalty. However, you will not get a negative total on any group of eight questions.

T F	If the 3×3 matrices A and B have three pivots each, then A can be transformed to B by means of elementary row operations.
T F	If A is an invertible matrix and $AB = AC$, then $B = C$.
T F	A linearly dependent collection of vectors in $\mathbf{R}^n$ must contain more than n vectors.
T F	The pivot columns of the row-reduced form of any matrix A form a basis for Col(A).
T F	The columns of an invertible $n \times n$ matrix form a basis of $\mathbf{R}^n$.
T F	If a finite set S of vectors spans a vector space V, then some subset of S is a basis of V.
T F	The dimension of the column space of a matrix is equal to the number of columns that do *not* contain pivots.
T F	If A is any $m \times n$ matrix, then the range of the linear map $\mathbf{x} \mapsto A\mathbf{x}$ is $\mathbf{R}^m$.

$m\left[\overset{n}{}\right] \; [n] \to \; \left[\,n\right.$

T F	A change-of-coordinates matrix is always invertible.
T F	If no vector in the set $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ in $\mathbf{R}^3$ is a multiple of one of the other vectors, then S is linearly independent.
T F	If the linear system $A\mathbf{x} = \mathbf{b}$ has more than one solution for some value of $\mathbf{b}$, then so does the linear system $A\mathbf{x} = \mathbf{0}$.
T F	The rank of a matrix is defined as the dimension of its nullspace.
T F	If A is an invertible $n \times n$ matrix, then the system $A\mathbf{x} = \mathbf{b}$ is consistent for *each* $\mathbf{b}$ in $\mathbf{R}^n$.
T F	The columns of any 5×4 matrix are linearly dependent.
T F	The dimensions of the row space and of the column space of a matrix A agree, whether or not A is square.
T F	The second column of AB is the second column of A multiplied on the right by B.
T F	When two linear transformations are performed in succession, the combined effect is not always a linear transformation.
T F	The non-zero rows of a matrix A form a basis of its row space.
T F	The range of a linear map between vector spaces, $T : V \to W$, is a linear subspace of W.
T F	If A, B are $n \times n$ matrices and $AB = BA$, then $(A + B)(A - B) = A^2 - B^2$.
T F	If A, B are square matrices, then $(AB)^T = A^T B^T$. $\quad AA + \cancel{BA} - \cancel{BA} - BB$
T F	If two matrices have the same reduced row echelon form, then their column spaces agree.
T F	A linear map preserves the operations of vector addition and scalar multiplication.
T F	If the matrix A is invertible, then A^{-1} is also invertible, and its inverse is A.

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$

$\to \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \begin{pmatrix} 5 & 7 \\ 6 & 8 \end{pmatrix}' \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} = \begin{bmatrix} 19 & 43 \\ 22 & 50 \end{bmatrix}$

36

Question 2. (12 pts)
For the following matrix, find bases for the row space, column space, nullspace and left nullspace. Describe your procedure clearly.

$$A = \begin{bmatrix} 1 & 2 & 2 & 3 \\ 4 & 3 & 5 & 4 \\ 3 & 1 & 3 & 1 \end{bmatrix}$$

Row space →
$$\begin{bmatrix} 1 & 2 & 2 & 3 \\ 4 & 3 & 5 & 4 \\ 3 & 1 & 3 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 2 & 3 \\ 0 & -5 & -3 & -8 \\ 0 & -5 & -3 & -8 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 4/5 & -1/5 \\ 0 & 1 & 3/5 & 8/5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\boxed{\text{basis Row } A = \{[1\ 0\ 4/5\ -1/5], [0\ 1\ 3/5\ 8/5]\}}$$ ✓

Col space →

$$\boxed{\text{basis Col } A = \left\{\begin{bmatrix}1\\4\\3\end{bmatrix}, \begin{bmatrix}2\\3\\1\end{bmatrix}\right\}}$$ ✓

Nul space →
$$\begin{bmatrix} 1 & 0 & 4/5 & -1/5 \\ 0 & 1 & 3/5 & 8/5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{Nul } A = x_3\begin{bmatrix}-4/5\\-3/5\\1\\0\end{bmatrix} + x_4\begin{bmatrix}1/5\\-8/5\\0\\1\end{bmatrix}$$

$$\boxed{\text{basis Nul } A = \left\{\begin{bmatrix}-4/5\\-3/5\\1\\0\end{bmatrix}, \begin{bmatrix}1/5\\-8/5\\0\\1\end{bmatrix}\right\}}$$ ✓

LNull space
$$\left[\begin{array}{cccc|ccc} 1 & 2 & 2 & 3 & 1 & 0 & 0 \\ 4 & 3 & 5 & 4 & 0 & 1 & 0 \\ 3 & 1 & 3 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cccc|ccc} 1 & 2 & 2 & 3 & 1 & 0 & 0 \\ 0 & -5 & -3 & -8 & -4 & 1 & 0 \\ 0 & -5 & -3 & -8 & -3 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cccc|ccc} 1 & 2 & 2 & 3 & 1 & 0 & 0 \\ 0 & -5 & -3 & -8 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 \end{array}\right]$$

basis LNull $A = \{[1\ -1\ 1]\}$ ✓

also
$$\begin{bmatrix} 1 & 4 & 3 \\ 2 & 3 & 1 \\ 2 & 5 & 3 \\ 3 & 4 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 3 \\ 0 & -5 & -5 \\ 0 & -3 & -3 \\ 0 & -8 & -8 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \to x_3\begin{bmatrix}1\\-1\\1\end{bmatrix}$$ ✓

Question 3. (12 pts)
For each of the following matrices, find the inverse, or explain why it is not invertible:

a. $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$; b. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}$; c. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 5 & 3 & 0 \end{bmatrix}$; d. $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 1 & 1 & a & 1 & 1 \\ 10 & 9 & 8 & 7 & 6 \\ 5 & 4 & 3 & 2 & 1 \end{bmatrix}$, where $a \in \mathbf{R}$.

a) $\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -1 & -2 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & 0 & -3 & 2 \\ 0 & 1 & 2 & -1 \end{array}\right]$

$a^{-1} = \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}$

3

b.) b. is not square ⇒ not invertable → cannot row reduce to the identity matrix.

3

c.) $\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 3 & 4 & 0 & 1 & 0 \\ 5 & 3 & 0 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -2 & -2 & 1 & 0 \\ 0 & -7 & -15 & -5 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & -1 & 0 \\ 0 & 0 & -1 & 9 & -7 & 1 \end{array}\right]$

$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & -39 & 30 & -4 \\ 0 & 1 & 0 & 20 & -15 & 2 \\ 0 & 0 & 1 & -9 & 7 & -1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -12 & 9 & -1 \\ 0 & 1 & 0 & 20 & -15 & 2 \\ 0 & 0 & 1 & -9 & 7 & -1 \end{array}\right]$.

$c^{-1} = \begin{bmatrix} -12 & 9 & -1 \\ 20 & -15 & 2 \\ -9 & 7 & -1 \end{bmatrix}$

3

d) d is not invertable ⇒ when $a = 1$

→ row 1 = row 2 − 5(row 3)

$[1\ 2\ 3\ 4\ 5] = [6\ 7\ 8\ 9\ 10] - [5\ 5\ 5\ 5\ 5]$

$= [1\ 2\ 3\ 4\ 5]$

1

⇒ d will not row reduce to the identity matrix

Question 4. (15 pts, 5+5+5)

Let P_3 be the vector space of polynomials of degree no more than 3. Define a map $T : P_3 \to P_3$ as follows: the polynomial $p(x) \in P_3$ is sent to the polynomial

$$(Tp)(x) = x^2 \cdot p(1) + xp'(x) - p''(x).$$

For instance, the constant polynomial 1 is sent to x^2, while the polynomial x^2 is sent to $3x^2 - 2$.

- a Write down the definition of a linear map, and verify that T is a linear map.
- b Find the matrix representing T in the basis $\{1, x, x^2, x^3\}$ of P_3.
- c Find a polynomial $q(x)$ such that $(Tq)(x) = x^2 + x + 1$.

a.) A linear map is a map between two linear spaces that preserves vector addition and scalar multiplication and maps the zero vector to the zero vector.

$$T(p_1(x)+p_2(x)) = x^2(p_1(1)+p_2(1)) + x(p_1'(x)+p_2'(x)) - (p_1''(x)+p_2''(x))$$
$$= T(p_1(x)) + T(p_2(x))$$

derivatives can be broken w/ into sums

$$T(p_1(ax)) = ax^2 p_1(1) + ax\, p_1'(x) - ap''(x) = aT(p_1(x))$$
$$T(0) = x^2(0) + 0 + 0 = 0$$

can pull a scalar out of a derivative

b.)

$1 \to x^2$

$x \to x^2 + x$

$x^2 \to 3x^2 - 2$

$x^3 \to x^2 + 3x^2(x) - 6x = 3x^3 + x^2 - 6x$

$$\boxed{T = \begin{bmatrix} 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & -6 \\ 1 & 1 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}}$$

c

$$\left[\begin{array}{cccc|c} 0 & 0 & -2 & 0 & 1 \\ 0 & 1 & 0 & -6 & 1 \\ 1 & 1 & 3 & 1 & 1 \\ 0 & 0 & 0 & 3 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 1 & 3 & 1 & 1 \\ 0 & 1 & 0 & -6 & 1 \\ 0 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 3 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 3 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1/2 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 3/2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1/2 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 3/2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1/2 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$\boxed{q(x) = \frac{3}{2} + x - \frac{x^2}{2}}$$

Check: $x^2(p(1)) + xp'(x) - p''(x)$

$= x^2\left(\frac{3}{2} + \frac{1}{2}\right) + x(1 - x) - (-1)$

$= 2x^2 + x - x^2 + 1 = x^2 + x + 1$