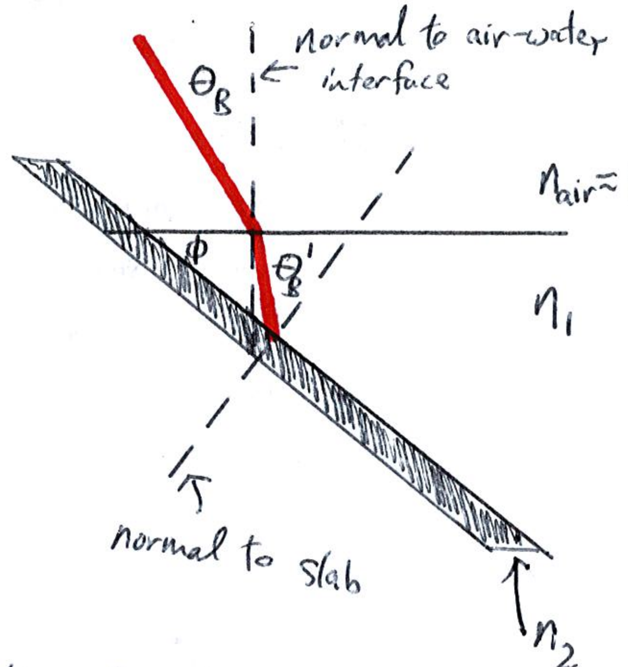
Midterm 2 Solutions

(1) There are two possible orientations for the slab.
I will show both:

Orientation 1



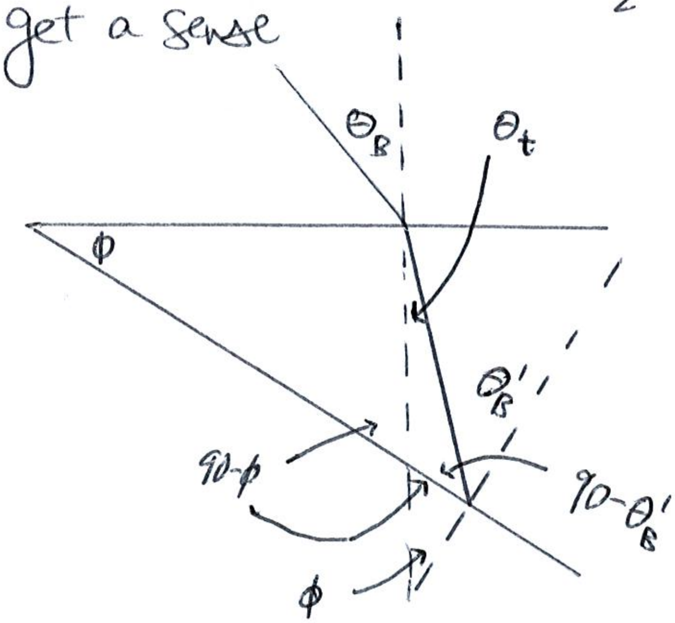
We know that $\tan\theta_B = \frac{n_1}{n_{air}} \doteq n_1$.

and also that $\tan\theta_B' = \frac{n_2}{n_1}$.

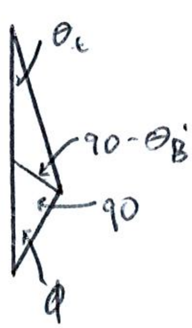
From Snell's law, we have that

$$n_{air}\sin\theta_B = n_1\sin\theta_t$$

Let's blow up the image so we get a sense of the geometry:



From the figure, we can see the following triangle



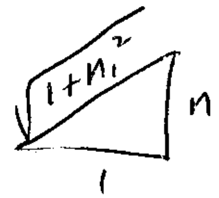
The sum of the angles is 180:

$$180 = \phi + 90 + 90 - \theta_B' + \theta_t$$

So $\quad \phi = \theta_B' - \theta_t$

$$= \arctan\left(\frac{n_2}{n_1}\right) - \arcsin\left(\frac{\sin\theta_B}{n_1}\right)$$

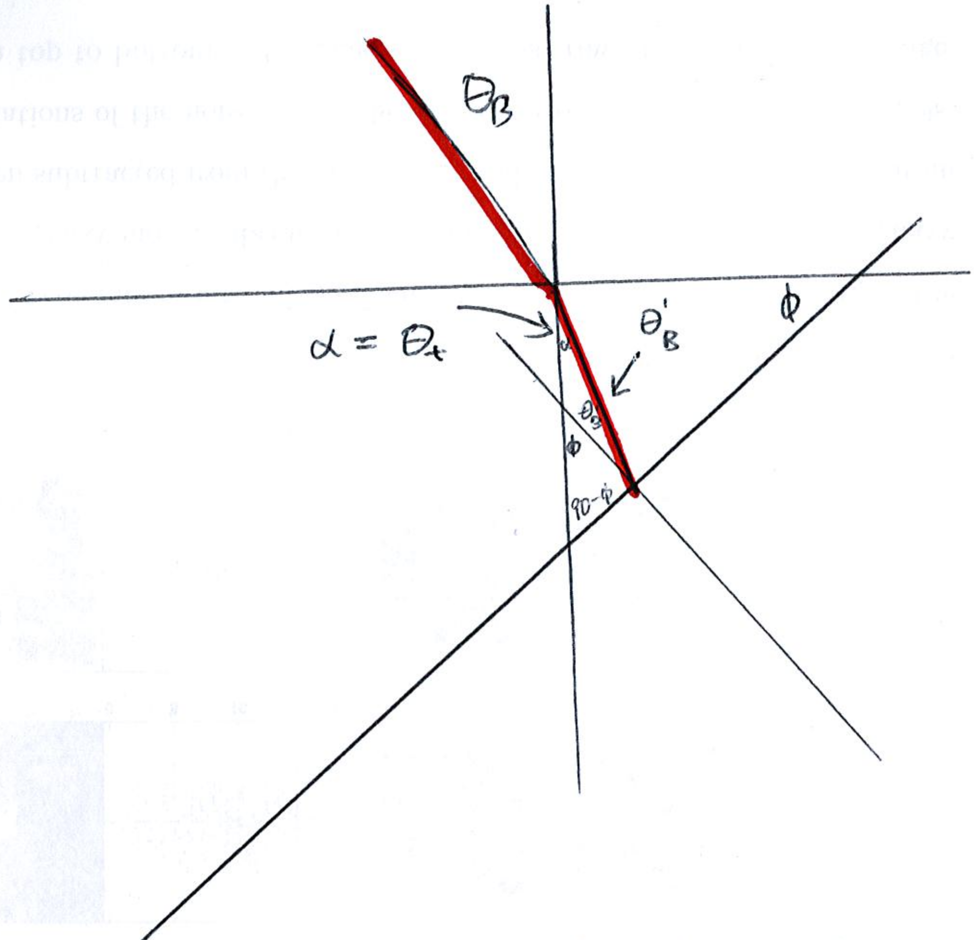
We can use trig to find $\sin\theta_B$: $\quad \tan\theta_B = n_1$



$$\sin\theta_B = \sin\left(\arctan\left(n_1\right)\right) = \frac{n_1}{\sqrt{1+n_1^2}}$$

So $\quad \boxed{\phi = \arctan\left(\frac{n_2}{n_1}\right) - \arcsin\left(\frac{1}{\sqrt{1+n_1^2}}\right)}$

Orientation 2



again, sum of angles gives: $\alpha + \theta_B' + \cancel{90} + 90 - \phi = \cancel{180}$

$$\alpha + \theta_B' = \phi$$

So from before, $\alpha = \arcsin\left(\frac{1}{\sqrt{1+n_1^2}}\right) = \theta_t$

$$\theta_B' = \arctan\left(\frac{n_2}{n_1}\right)$$

So $$\boxed{\phi = \arctan\left(\frac{n_2}{n_1}\right) + \arcsin\left(\frac{1}{\sqrt{1+n_1^2}}\right)}$$

(2) Diffraction minima obey $D \sin\theta = m\lambda$, $m = 1, 2, 3, \ldots$

Interference maxima obey $d \sin\theta = n\lambda$, $n = 1, 2, 3, \ldots$

where D is slit width, d is slit separation.

In this problem, $d = 2D$.

Divide the two equations to get $\frac{D}{d} = \frac{m}{n}$.

So $\frac{1}{2} = \frac{m}{n}$ or $\underline{n = 2m}$. So when $m = 1$, corresponding to the first diffraction minimum, $n = 2$, the second interference maximum (not counting the central max).

So all even-numbered interference maxima will be missing, i.e. $\{2, 4, 6, 8, \ldots\}$.

(3) Let's switch to the frame of the flying slab:

Light travels at speed c in vacuum and speed $v = \frac{c}{n}$ in the slab.
The total distance between A and B in this frame is $L' = \frac{L}{\gamma}$.

So
$$\underbrace{\Delta t}_{\text{Time in moving frame}} = \underbrace{\frac{\frac{L}{\gamma} - d}{c}}_{\text{Amount traveled outside slab}} - \underbrace{\frac{v\Delta t}{c}}_{\text{Amount that B has moved}} + \underbrace{\frac{d}{c/n}}_{\text{amount traveled in slab.}}$$

So
$$\Delta t\left(1 + \frac{v}{c}\right) = \frac{1}{c}\left(\frac{L}{\gamma} - d + dn\right)$$

$$\Delta t = \frac{1}{c+v}\left(\frac{L}{\gamma} - d + dn\right)$$

This is in the stationary frame of the slab. In the other frame
$$\boxed{\Delta t' = \gamma \Delta t = \frac{\gamma}{(c+v)}\left(L/\gamma - d + dn\right)}$$

(b) For $v = 0$, $\gamma = 1$, so

$$\Delta t' = \underline{\frac{1}{c}\left(\underbrace{L/\gamma - d}_{\text{amount travelled outside slab}} + \underbrace{dn}_{\text{amount traveled in slab}}\right)}$$

So this gives the amount of time in the stationary frame where nothing is moving.

(c) For $n = 1$,

$$\Delta t' = \frac{\gamma}{(c+v)}\left(L/\gamma + d - d\right) = \frac{\gamma\,(L/\gamma)}{(c+v)}$$

$$\underline{\Delta t' = \frac{L}{c+v}}$$ which is what we expect for light towards a moving object.

(4) (a) The events must be spacelike meaning that

$$(\Delta S)^2 = (\Delta x)^2 - (c\Delta t)^2 > 0.$$

$$\text{so } (\Delta x)^2 > (c\Delta t)^2.$$

(b) To find proper length note that for any two frames, S and S',

$(x)^2 - (c\Delta t)^2 = (x')^2 - (c\Delta t')^2$. For proper length $\Delta t = 0$

so

$$x_{prop} = \sqrt{(x')^2 - (c\Delta t')^2}$$

In this problem, we're given Δx and Δt in frame S, so

$$\Delta x_p = \sqrt{(\Delta x)^2 - (c\Delta t)^2}.$$

By length contraction, $\frac{\Delta x_p}{\gamma} = \Delta x$, where $\gamma = \frac{1}{\sqrt{1 - \frac{w^2}{c^2}}}$

so

$$1 - \frac{w^2}{c^2} = \frac{\Delta x^2}{\Delta x_p^2}$$

$$\frac{w}{c} = \sqrt{1 - \frac{\Delta x^2}{\Delta x_p^2}}$$

(c) We know $x' = \gamma(x - vt)$ and $t' = \gamma(t - \frac{vx}{c^2})$

for any two ~~...~~ frames S and S'.

So $\Delta x' = \gamma(\Delta x - v\Delta t)$

$\Delta t' = \gamma(\Delta t - \frac{v\Delta x}{c^2})$,

If the order of flashes reversed then $\Delta t' = -\Delta t$, but $\Delta x' = \Delta x$.

So $\frac{\Delta x'}{\Delta t'} = \frac{-\Delta x}{\Delta t} = \frac{\Delta x - v\Delta t}{\Delta t - \frac{v\Delta x}{c^2}}$ which we can rewrite as

$$-\Delta x \Delta t + \frac{v\Delta x^2}{c^2} = \Delta x \Delta t - v\Delta t^2$$

and $v\left(\frac{\Delta x^2}{c^2} + \Delta t^2\right) = 2\Delta x \Delta t$

So $$v = \frac{2\Delta x \Delta t}{\left(\frac{\Delta x^2}{c^2} + \Delta t^2\right)}$$

Since $\Delta x > 0$ and $\Delta t > 0$,

$v > 0$

so positive x-direction

⑤ $\pi^+ \rightarrow \mu^+ + \nu$

We are in the COM frame, so net momentum is zero.

The conservation equation is: $P_\pi = P_\mu + P_\nu$ where these are energy-momentum 4-vectors. Since the pion is at rest,

$$P_\pi = \begin{pmatrix} m_\pi c \\ 0 \end{pmatrix} \begin{array}{l} \leftarrow E/c \\ \leftarrow p_x \end{array}$$

. also, since $m_\nu \cong 0$ we can write

$E_\nu/c = p_x$ using $E = \sqrt{p^2c^2 + m^2c^4}$

So $P_\nu = \begin{pmatrix} p_x \\ p_x \end{pmatrix}$. Finally, $P_\mu = \begin{pmatrix} E/c \\ -p_x \end{pmatrix}$.

So

$$\begin{pmatrix} m_\pi c \\ 0 \end{pmatrix} = \begin{pmatrix} p_x \\ p_x \end{pmatrix} + \begin{pmatrix} E/c \\ -p_x \end{pmatrix} \begin{array}{l} \leftarrow \text{Cons. of Energy} \\ \leftarrow \text{cons. of momentum.} \end{array}$$

So using $p_x = \sqrt{E_\mu^2/c^2 - m_\mu^2 c^2}$ we can write

$$m_\pi c - E/c = \sqrt{E^2/c^2 - m_\mu^2 c^2}$$ and squaring,

$$m_\pi^2 c^2 - 2m_\pi E + E^2/c^2 = E^2/c^2 - m_\mu^2 c^2$$

$$E = \frac{m_\pi^2 c^2 + m_\mu^2 c^2}{2m_\pi}.$$

Remember that $E = m_\mu c^2 \gamma$ so

$$\gamma = \frac{m_\pi^2 c^2 + m_\mu^2 c^2}{2 m_\pi m_\mu c^2} = \frac{m_\pi^2 + m_\mu^2}{2 m_\pi m_\mu}$$

But $\gamma = \frac{1}{\sqrt{1-\beta^2}}$ so $\beta = \sqrt{1 - \frac{1}{\gamma^2}}$

Thus $$\boxed{\beta = \sqrt{1 - \left(\frac{4 m_\pi m_\mu}{m_\pi^2 + m_\mu^2}\right)^2}}$$