SOLUTIONS

MT1.1 (30 Points) Each of the following makes a statement. Indicate whether that statement is true or false. If you are unsure of your answer, feel free to provide justification. If there is no justification, then your grade will be all or nothing.

(a) True or False: Given a function F, where

$$F\colon [\mathbb{N}_0 \to \mathbb{N}_0] \to [\mathbb{N}_0 \to \mathbb{N}_0]$$

is given by

$$\forall\, x \in [\mathbb{N}_0 \to \mathbb{N}_0],\ \forall\, n \in \mathbb{N}_0, \quad (F(x))(n) = 5,$$

then the system is memoryless.

Solution: True.

(b) True or False: Given a function F, where

$$F \in [\mathbb{N}_0 \to [\mathbb{N}_0 \to \mathbb{N}_0]],$$

then the following is well-formed and true:

$$(F(5))(5) \in \mathbb{N}_0.$$

Solution: True.

(c) True or False: The following assertion is well-formed and true:

$$[\mathbb{N}_0 \to \mathbb{R}] \subset [\mathbb{N}_0 \to \mathbb{N}_0].$$

Solution: False.

(d) True or False: Given a function F, where

$$F\colon [\mathbb{N}_0 \to \mathbb{N}_0] \to [\mathbb{N}_0 \to \mathbb{N}_0]$$

is given by

$$\forall\, x \in [\mathbb{N}_0 \to \mathbb{N}_0],\ \forall\, n \in \mathbb{N}_0, \quad (F(x))(n) = n,$$

then the system is memoryless.

Solution: False.

(e) True or False: Consider a state machine whose state at time n is a 2×1 column vector $s(n) \in \mathbb{Z}^2$. Suppose the input space is $\{0,1\}$. That is, at time n, the input value is $x(n) \in \{0,1\}$. Suppose the at each reaction the state is updated according to the following equation:

$$s(n+1) = As(n) + bx(n),$$

where

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{and} \quad s(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Then this system will only ever be in a finite number of possible states. (That is, there is an equivalent finite state machine).

Solution: True.

(f) True or False: Consider a variant of the previous question where everything is the same except that

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Then this system will only ever be in a finite number of possible states. (That is, there is an equivalent finite state machine).

Solution: True.

MT1.2 (40 Points) Consider three functions:

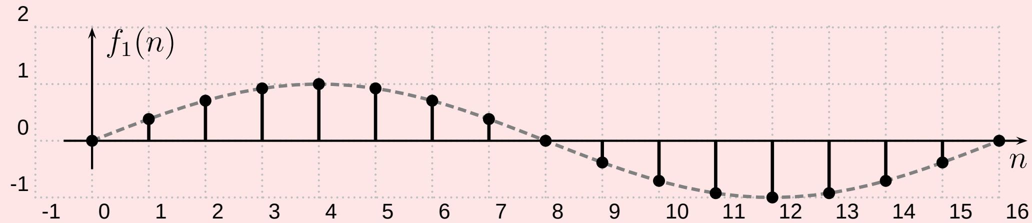
$$\begin{aligned} f_1 \colon \mathbb{N}_0 \to \mathbb{R}, &\quad \forall n \in \mathbb{N}_0, \quad f_1(n) = \sin(\pi n/8) \\ f_2 \colon \mathbb{N}_0 \to \mathbb{N}_0, &\quad \forall n \in \mathbb{N}_0, \quad f_2(n) = \lfloor n/2 \rfloor \\ f_3 \colon \mathbb{N}_0 \to \mathbb{N}_0, &\quad \forall n \in \mathbb{N}_0, \quad f_3(n) = 2n \end{aligned}$$

where $\lfloor x \rfloor$ is the "floor" operator, denoting for any $x \in \mathbb{R}$ the greatest integer m such that $m \leq x$.

For the sketches below, label the salient points. Your sketch need not be numerically accurate (a rough estimate of the values of the cosine is adequate, except where the values are easy to know exactly). Be sure that your sketch renders all interesting features of the function. In case you have forgotten your trigonometry, a few of the key salient points of the cosine function are $\sin(0) = 0$, $\sin(\pi/2) = 1$, $\sin(\pi) = 0$, $\sin(3\pi/2) = -1$, etc.

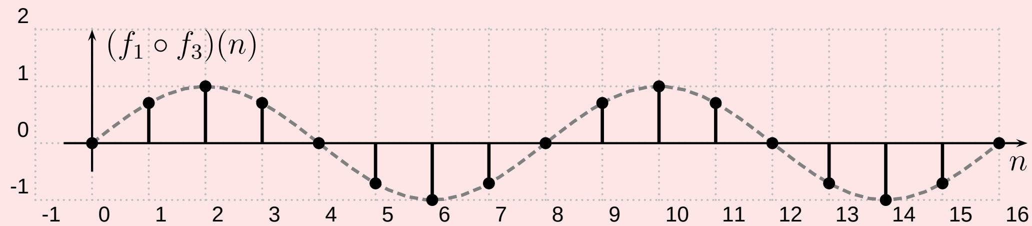
(a) Sketch the graph of f_1 over the subset $\{0, 1, 2, \cdots, 16\}$ of the domain.

Solution:



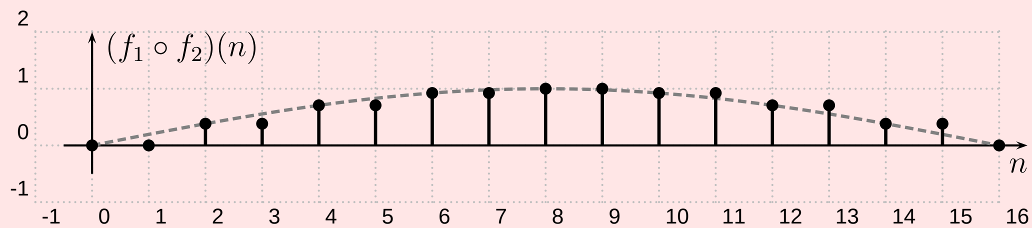
(b) Sketch the graph of $f_1 \circ f_3$ over the same subset of the domain.

Solution:



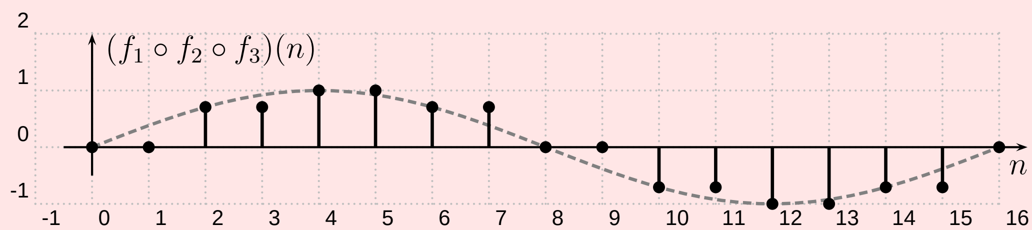
(c) Sketch the graph of $f_1 \circ f_2$ over the same subset of the domain.

Solution:



(d) Sketch the graph of $f_1 \circ f_3 \circ f_2$ over the same subset of the domain.

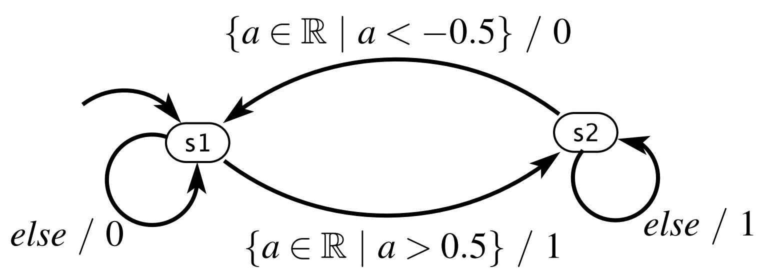
Solution:



MT1.3 (40 Points) Consider a system $F\colon X \to Y$, where

$$\begin{aligned} X &= [\mathbb{N}_0 \to \mathbb{R}] \\ Y &= [\mathbb{N}_0 \to \{0,1\}], \end{aligned}$$

where the function F is defined by the following state machine:



The initial state is s1.

(a) Suppose that the input $x \in X$ is given by, $\forall\, n \in \mathbb{N}_0$,

$$x(n) = 0.25n.$$

Give the trace and the output sequence over $0 \le n \le 5$ (the first six reactions).

Solution:

$$s1 \xrightarrow{0} s1 \xrightarrow{0.25} s1 \xrightarrow{0.5} s1 \xrightarrow{0.75} s2 \xrightarrow{1} s2 \xrightarrow{1.25} s2 \cdots$$

The output sequence is

$$(0,0,0,1,1,1).$$

(b) Now, instead suppose that the input $x \in X$ is given by, $\forall\, n \in \mathbb{N}$,

$$x(n) = cos(\pi n).$$

Give the trace and the output sequence over $0 \leq n \leq 5$ (the first six reactions).

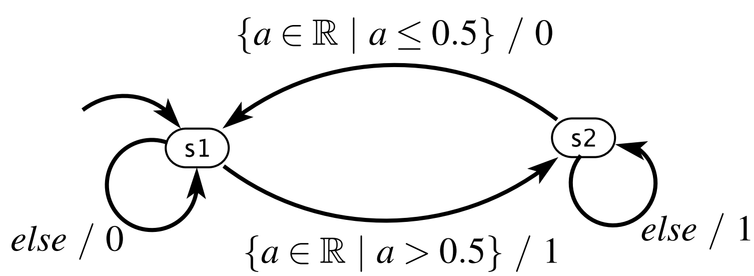
Solution:

$$s1 \xrightarrow{1} s2 \xrightarrow{-1} s1 \xrightarrow{1} s2 \xrightarrow{-1} s1 \xrightarrow{1} s2 \xrightarrow{-1} s1 \cdots$$

(c) Describe in words how the output is related to the input, for arbitrary input. Under what circumstances is the output 1 or 0?

Solution: The output is 1 if the input has previously been greater than 0.5, and since the last time it was greater than 0.5, it has not dipped below -0.5. The output is 0 if either the input has never risen above 0.5, or if the input has previously been less than-0.5, and since the last time it was less than -0.5, it has not risen above 0.5.

(d) Consider the following variant, where the guard on the transition from s2 to s1 has changed:



Is this system memoryless? If so, give the function f (including defining its domain and range) such that for all $n \in \mathbb{N}_0$ and $x \in X$, $(F(x))(n) = f(x(n))$.

Solution: The system is memoryless. The function $f\colon \mathbb{R} \to \{0,1\}$ is given by, for all $t \in \mathbb{R}$,

$$f(t) = \begin{cases} 1 & \text{if } t > 0.5 \\ 0 & \text{otherwise} \end{cases}$$