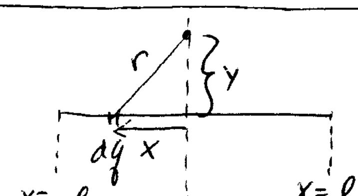
Huang Midterm 2 Problem #1

30 points total

I would like to point out common errors that occurred throughout this problem:

- Electric field is a vector field and when adding contributions from different charges, the contributions must be added as vectors (i.e. component-by-component)
- Electric potential is a scalar field, so it has no components (it is simply a number assigned to each point in space). Thus adding contributions from different charges follows the rules for adding ordinary numbers.
- The formula $V = -\int \vec{E}\cdot d\vec{\ell}$ is more correctly written as $V(\vec{r}_2) - V(\vec{r}_1) = -\int_{\vec{r}_1}^{\vec{r}_2} \vec{E}\cdot d\vec{\ell}$. If we pick our reference point at ∞ (assumed in this problem), we have $V(\infty) - V(\vec{r}_1) = -\int_{\vec{r}_1}^{\infty} \vec{E}\cdot d\vec{\ell}$. By definition of a reference point, $V(\infty)=0$, and so we have $V(\vec{r}_1) = \int_{\vec{r}_1}^{\infty} \vec{E}\cdot d\vec{\ell}$. Notice that this is not an integral over a charge distribution. To add up contributions of small charges, you want to use $V = \int_{\substack{\text{charge} \\ \text{distribution}}} \frac{dq}{4\pi\varepsilon_0 r}$ which already assumes $V(\infty)=0$

(a) (4 pts) $V(x=0, y) = \int_{\text{rod}} \frac{dq}{4\pi\varepsilon_0 r}$



$$V(x=0,y) = \int_{-\ell}^{\ell} \frac{dq}{4\pi\varepsilon_0 \sqrt{x^2+y^2}}$$

$$dq = \lambda\, dx = ax\, dx$$

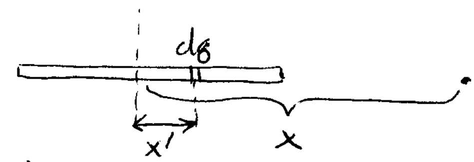
$$V(x=0,y) = \frac{a}{4\pi\varepsilon_0} \int_{-\ell}^{\ell} \frac{x\,dx}{\sqrt{x^2+y^2}} \quad \text{(2pts)}$$

$\boxed{V(x=0,y) = 0}$ (2pts) because the integrand is an odd function of x, while the integration interval is symmetric in x

Common mistakes:

- Gauss's law cannot be used to find the Electric field because the symmetry is lacking
- When asked for the potential at some point it is much easier to find it directly rather than going through electric field.
- $dq = \lambda dx$ is the defining formula for λ. $Q_{tot} = \lambda L_{tot}$ is a special case of the above formula for λ = constant.

(b) (8pts) $V(x>\ell, y=0) = \int_{rod} \frac{dq}{4\pi\varepsilon_0 r}$



$r = x - x'$, $dq = \lambda\, dx'$ (notice the prime)
$= a x' dx'$

x' ranges from $-\ell$ to $+\ell$, while x is just a fixed observation point

$$V(x>\ell, y=0) = \int_{-\ell}^{\ell} \frac{a}{4\pi\varepsilon_0} \frac{x'dx'}{x-x'} \quad (6\text{pts})$$

We can perform the substitution $u = x - x'$ $\quad du = -dx'$
$x' = x - u$

We will also have to change the limits to $u(-\ell)$, $u(\ell)$

$$V(x>\ell, y=0) = -\frac{a}{4\pi\varepsilon_0} \int_{x+\ell}^{x-\ell} \frac{(x-u)\,du}{u} = \frac{a}{4\pi\varepsilon_0}\left(-\int_{x+\ell}^{x-\ell} \frac{x\,du}{u} + \int_{x+\ell}^{x-\ell} du\right)$$

$$= \frac{a}{4\pi\varepsilon_0}\left(x\int_{x-\ell}^{x+\ell} \frac{du}{u} - \int_{x-\ell}^{x+\ell} du\right)$$

$$= \frac{a}{4\pi\varepsilon_0}\left(x \ln u \Big|_{x-\ell}^{x+\ell} - u\Big|_{x-\ell}^{x+\ell}\right)$$

$$\boxed{V(x>\ell, y=0) = \frac{a}{4\pi\varepsilon_0}\left(x \ln\left(\frac{x+\ell}{x-\ell}\right) - 2\ell\right) \quad (2\text{pts})}$$

Common mistakes:

- There was general confusion as to what was being integrated with what limits. Hopefully the solutions are clear enough

(6 pts)

(c) We want leading-order behavior of $V(x,y=0)$ for large x. There are two good ways of Taylor expanding. The easier one is to start with the integral expression for V:

$$V(x,y=0) = \int_{-\ell}^{\ell} \frac{a}{4\pi\varepsilon_0} \frac{x'dx'}{x-x'} = \frac{a}{4\pi\varepsilon_0} \int_{-\ell}^{\ell} \frac{x'}{x} \frac{1}{1-x'/x} dx'$$

$$= \frac{a}{4\pi\varepsilon_0} \int_{-\ell}^{\ell} \frac{x'}{x} \left(1-\frac{x'}{x}\right)^{-1} dx'$$

Since x is large, while x' is bounded by ℓ, we have $\frac{x'}{x} \ll 1$, so $\left(1-\frac{x'}{x}\right)^{-1} \approx 1+\frac{x'}{x}$

$$V(x\gg\ell, y=0) \approx \frac{a}{4\pi\varepsilon_0} \int_{-\ell}^{\ell} \frac{x'}{x}\left(1+\frac{x'}{x}\right) dx' \quad (4\text{pts})$$

$$= \frac{a}{4\pi\varepsilon_0} \left[\int_{-\ell}^{\ell} \frac{x'dx'}{x} + \int_{-\ell}^{\ell} \frac{x'^2}{x^2} dx' \right]$$

The first term vanishes because it is an odd function of x'.

$$V(x\gg\ell, y=0) = \frac{a}{4\pi\varepsilon_0} \frac{1}{x^2} \int_{-\ell}^{\ell} x'^2 dx' = \frac{a}{4\pi\varepsilon_0} \frac{1}{x^2} \frac{1}{3} x'^3 \Big|_{-\ell}^{\ell}$$

$$\boxed{V(x\gg\ell, y=0) = \frac{a}{6\pi\varepsilon_0} \frac{\ell^3}{x^2} \quad (2\text{pts})}$$

The second method is to Taylor expand the final expression for V in part b, and go to 3rd order:

$$V(x\gg\ell, y=0) = \frac{a}{4\pi\varepsilon_0} \left(x\left[\ln(x+\ell) - \ln(x-\ell)\right] - 2\ell \right)$$

$$= \frac{a}{4\pi\varepsilon_0} \left(x\left[\cancel{\ln x} + \ln\left(1+\frac{\ell}{x}\right) - \cancel{\ln x} - \ln\left(1-\frac{\ell}{x}\right)\right] - 2\ell \right)$$

$\frac{\ell}{x} \ll 1$, and $\ln(1+\alpha) \approx \alpha - \frac{1}{2}\alpha^2 + \frac{1}{3}\alpha^3$

(4 pts)

Thus $V(x\gg\ell, y=0) = \frac{a}{4\pi\varepsilon_0} \left(x\left[\frac{\ell}{x} - \cancel{\frac{1}{2}\left(\frac{\ell}{x}\right)^2} + \frac{1}{3}\left(\frac{\ell}{x}\right)^3 - \left(-\frac{\ell}{x}\right) + \cancel{\frac{1}{2}\left(\frac{\ell}{x}\right)^2} + \frac{1}{3}\left(\frac{\ell}{x}\right)^3 \right] - 2\ell \right)$

$$= \frac{a}{4\pi\varepsilon_0} \left(2\ell + \frac{2}{3}\frac{\ell^3}{x^2} - 2\ell \right) = \boxed{\frac{a}{6\pi\varepsilon_0} \frac{\ell^3}{x^2} \quad (2\text{pts})}$$

Common mistakes:

- $V(\infty)=0$ but in physics when we look at limiting behavior for large distances we are interested in non-zero leading order behavior. The expression you get should still fall to 0 at ∞.
- Taylor expanding V itself will not work because x is not small.
- Total charge of rod is 0, so there is no $\frac{1}{x}$ drop-off in the potential. However, the rod has a dipole moment, and hence there is a $\frac{1}{x^2}$ drop-off of the potential

(d) (6 pts) In part (a), we found that V along the y-axis is 0. The problem has cylindrical symmetry about the x-axis, so the entire $\boxed{\text{yz plane is an equipotential with } V=0}$ (2pts)

Since $\vec{E}=-\vec{\nabla}V$, $\vec{E}$ is perpendicular to equipotentials. (2pts)

Thus $\vec{E}$ points parallel to the x-axis. Since the positive charge is on the right (for $a>0$), and electric fields point from positive to negative, it follows that $\boxed{\vec{E} \text{ points in the } -\hat{x} \text{ direction.}}$ (2 pts)

Common mistakes:

- Using the result from part (b) to find $V(x=0)$ is incorrect because the result of part b is valid only for $x>l$.
- The equipotential surface is a 2-d surface of infinite extent, not a line or a disk
- The electric field does not point radially outward because it is not uniformly charged and not infinite

(e) (6 pts)

$\vec{E} = -\vec{\nabla} V$

All of the charges lie on the x-axis, and so the symmetry of the problem dictates that $E_y = E_z = 0$

Thus $\vec{E} = E_x \vec{x}$ and $E_x = -\frac{\partial V}{\partial x}$ (1pt)

$$V = \frac{q}{4\pi\epsilon_0}\left(x \ln\left(\frac{x+\ell}{x-\ell}\right) - 2\ell\right)$$

$$E_x = -\frac{\partial V}{\partial x} = -\frac{q}{4\pi\epsilon_0}\left(\ln\frac{x+\ell}{x-\ell} + x\left(\frac{1}{x+\ell} - \frac{1}{x-\ell}\right)\right) \text{ (2pts)}$$

Plug in $x = 2\ell$, $y = 0$

$$E_x = \frac{-q}{4\pi\epsilon_0}\left(\ln\frac{3\ell}{\ell} + 2\ell\left(\frac{1}{3\ell} - \frac{1}{\ell}\right)\right)$$

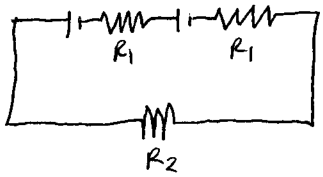
$$= -\frac{q}{4\pi\epsilon_0}\left(\ln 3 - \frac{4}{3}\right)$$

$$\boxed{\vec{E} = \frac{q}{4\pi\epsilon_0}\left(\frac{4}{3} - \ln 3\right)\vec{x}}$$ (2pts for magnitude, 1pt for direction)

Common mistakes:

- $\vec{\nabla}$ means apply the operator $\frac{\partial}{\partial x}\vec{x} + \frac{\partial}{\partial y}\vec{y} + \frac{\partial}{\partial z}\vec{z}$ to some function (in particular, it means differentiate)
- Differentiation takes place first, then you evaluate at $(2\ell, 0)$. Reversing the order always yields 0.
- Just because your answer to part (b) does not contain y does not mean $\frac{\partial V}{\partial y} = 0$. You found $V(x, y=0)$ and not V as a general function of y.
- Being on the x-axis does not guarantee $E_y = E_z = 0$

SECTION 2 – PROBLEM 2



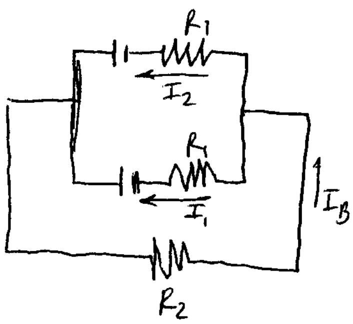
$V = IR$ +2

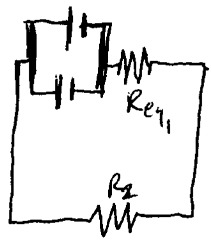
$R_{eq} = 2R_1 + R_2$ +2

$V = 2\mathcal{E}$ +2

6

a) $I = \frac{V}{R_{eq}} = \frac{2\mathcal{E}}{2R_1 + R_2}$





b) approach 1

$\mathcal{E} - I_1 R_1 - \mathcal{E} + I_2 R_1 = 0$ +2

$I_1 \cancel{R_1} = I_2 \cancel{R_1}$

JUNCTION RULE: $I_1 + I_2 = I_B$

$\therefore \frac{1}{2} I_B = I_1 = I_2$ +1

$\mathcal{E} - I_B R_2 - \frac{1}{2} I_B R_1 = 0$ +4

$I_B = \frac{\mathcal{E}}{R_2 + \frac{1}{2}R_1}$

approach 2

$V = IR$ +1

$R_{eq_1} = \frac{1}{2} R_1$ +1

$R_{eq} = \frac{1}{2} R_1 + R_2$ +3

$V = \mathcal{E}$ per loop +2

$I_B = \frac{\mathcal{E}}{\frac{1}{2} R_1 + R_2}$

7

c) $P_{A_{R_2}} = I_A^2 R_2 = \frac{4\mathcal{E}^2}{(2R_1 + R_2)^2} R_2$

+2 for I^2R or IV, $\frac{V^2}{R}$ (NO DOUBLE COUNTING)

+2 ANSWER

4

d) $P_{A_{R_2}} = P_{B_{R_2}}$ +1

$P_{B_{R_2}} = I_B^2 R_2 = \frac{4\mathcal{E}^2}{(2R_2 + R_1)^2} R_2$ +2

$\frac{4\mathcal{E}^2}{(2R_1 + R_2)^2} \cancel{R_2} = \frac{4\mathcal{E}^2}{(2R_2 + R_1)^2} \cancel{R_2}$

$2R_1 + R_2 = R_1 + 2R_2$ +3

$2\frac{R_1}{R_2} + 1 = \frac{R_1}{R_2} + 2$

$\frac{R_1}{R_2} = 1$

5

e) $R_1 > R_2$

$\therefore P_A < P_B$ +2

STATEMENT OR MATH +1

e.g. $2R_1$ in P_A denominator vs $1R_1$ in P_B

3

Huang problem # 3

April 10, 2011

(a) [**4 pts.**] The electric field points *radially inward* [**1 pt.**]. Since the charge distribution is cylindrically symmetric, we pick a cylinder of radius r for our Gaussian surface $\mathcal{S}$. Then

$$\oint_{\mathcal{S}} \vec{E} \cdot d\vec{a} = E \times (2\pi r L) = \frac{Q_{enc}}{\varepsilon_0} = \frac{-Q}{\varepsilon_0}$$
$$\Rightarrow \boxed{\vec{E}(r) = \frac{Q}{2\pi\varepsilon_0 r L}(-\hat{r}).} \tag{0.1}$$

(b) [**4 pts.**] By symmetry, it is easiest to integrate along the line $d\vec{l} = dr\hat{r}$. The potential difference V_0 between a and b is

$$V_0 = -\int_a^b \vec{E} \cdot d\vec{l} = +\frac{Q}{2\pi\varepsilon_0 L}\int_a^b \frac{dr}{r}$$
$$\Rightarrow \boxed{V_0 = \frac{Q}{2\pi\varepsilon_0 L}\ln\left(\frac{b}{a}\right).} \tag{0.2}$$

(c) [**4 pts.**] Now we integrate only to r instead of b. Note that to avoid confusion I have relabeled the integration variables by dummy variables r'. The answer depends on where you set the zero of potential, which wasn't specified, so your answer is correct if it differs from mine by only a constant. Here I set the zero of potential at the inner radius a:

$$V(r) = -\int_a^r \vec{E} \cdot d\vec{l} = +\frac{Q}{2\pi\varepsilon_0 L}\int_a^r \frac{dr'}{r'}$$
$$\Rightarrow \boxed{V(r) = \frac{Q}{2\pi\varepsilon_0 L}\ln\left(\frac{r}{a}\right).} \tag{0.3}$$

(d) [**4 pts.**]

$$\boxed{C = \frac{Q}{V_0} = \frac{2\pi\varepsilon_0 L}{\ln\left(\frac{b}{a}\right)}.} \tag{0.4}$$

(e) [**4 pts.**]

$$\boxed{U = \frac{1}{2}QV = \frac{Q^2}{4\pi\varepsilon_0 L}\ln\left(\frac{b}{a}\right).} \tag{0.5}$$

(f) **[4 pts.]** The energy density is $u = \frac{1}{2}\varepsilon_0 E^2$, which we need to integrate over all space where there is a nonzero electric field, namely the space between the two shells. Taking $dV = 2\pi r L dr$ we have

$$U = \frac{1}{2}\int E^2 dV = \frac{1}{2}\varepsilon_0 \int_a^b \frac{Q^2}{4\pi^2\varepsilon_0^2 L^2 r^2}(2\pi r L dr) \tag{0.6}$$

$$\Rightarrow \boxed{U = \frac{Q^2}{4\pi\varepsilon_0 L}\ln\left(\frac{b}{a}\right).} \tag{0.7}$$

This checks with our answer from (e) (as it should)!

(g) **[6 pts.]** The upper half is a half cylindrical capacitor C_1 with a dielectric connected in series with an ordinary half cylindrical capacitor C_2. (The capacitance of a half cylinder is half the capacitance of a full cylindrical capacitor.) The combination is connected in parallel with a half-cylindrical capacitor C_3, which again has half the capacitance of a full cylindrical capacitor. We have:

$$C_1 = \frac{\pi K \varepsilon_0 L}{\ln\left(\frac{a+t}{a}\right)}, \tag{0.8}$$

$$C_2 = \frac{\pi\varepsilon_0 L}{\ln\left(\frac{b}{a+t}\right)}, \tag{0.9}$$

$$C_3 = \frac{\pi\varepsilon_0 L}{\ln\left(\frac{b}{a}\right)} \tag{0.10}$$

The equivalent capacitance is

$$C_{eq} = \left(\frac{1}{C_1} + \frac{1}{C_2}\right)^{-1} + C_3 \tag{0.11}$$

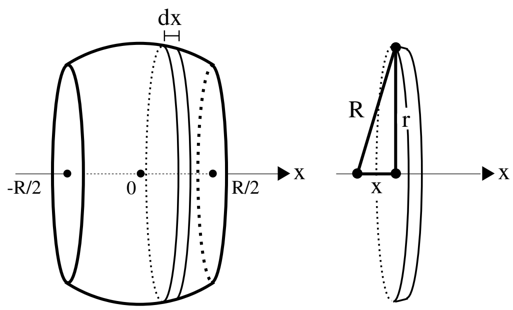
$$\Rightarrow \boxed{C_{eq} = \frac{\pi\varepsilon_0 L}{\frac{1}{K}\ln\left(\frac{a+t}{a}\right) + \ln\left(\frac{b}{a+t}\right)} + \frac{\pi\varepsilon_0 L}{\ln\left(\frac{b}{a}\right)}.} \tag{0.12}$$

Common mistakes: (i) I took a point off for writing $\ln r$ instead of $\ln\left(\frac{r}{a}\right)$ or $\ln\left(\frac{r}{b}\right)$, since the quantity inside the logarithm should be dimensionless.

(ii) A number of people got part (a) wrong, finding the electric field to be constant or go like $1/r^2$, and had this mistake propagate through. I gave partial credit for later answers if your approach got you the correct answer given your incorrect answer from (a). In the case that the student found the electric field to be constant, I gave lower fractions of the points in later parts since this made the calculations a lot easier. In the case that there was just a factor of two or some non-fundamental mistake that propagated, I tried to take off points only once (unless you made some other mistake).

(iii) The most common mistake was integrating $u = \frac{1}{2}\varepsilon_0 E^2$ over dr (or some even more creative things) instead of $dV = 2\pi r L dr$ for part (f). u is energy per volume, so be careful about your units!

Problem 4 [15 pts]



We cannot use any formula directly (such as $R = \rho l/A$) to calculate the resistance of this geometry directly since the cross sectional area changes as the current moves inside the conductor. But we can break the truncated sphere into tiny discs, each with resistance dR, assume that the current flows uniformly through these discs, and calculate the total resistance R integrating.

So, for each tiny disc, we have:

$$dR = \frac{\rho\, dl}{A}$$

where dl is the thickness of the disk, and A is the cross-sectional area that the current "sees" as it goes from left to right. According to the geometry above, we have that $dl = dx$, $A = \pi\, r^2 = \pi\,(R^2 - x^2)$. The integration should go from $x = -R/2$ to $x = R/2$.

$$R = \int dR = \int_{-R/2}^{R/2} \frac{\rho\, dx}{\pi\,(R^2 - x^2)} = \frac{2\rho}{\pi} \int_0^{R/2} \frac{dx}{(R^2 - x^2)}$$

Using the given table of integrals, we get:

$$R = \frac{2\rho}{\pi} \times \frac{1}{2R} \ln\left(\frac{R+x}{R-x}\right)_{x=0}^{X=R/2} = \frac{\rho}{\pi R}\left[\ln(3) - \ln(1)\right]$$

$$\boxed{R = \frac{\rho}{\pi R} \ln 3}$$

Rubrics & Common Errors

You got:

- [6 pts] if you realized that you should break the resistors into tiny pieces, and wrote that $dR = \frac{\rho\, dl}{A}$, but not $dR = \frac{\rho\, l}{dA}$, $dR = \frac{\rho\, dl}{dA}$, or any other wild variation;
- [5 pts] if you have shown a clear understanding of the physics and geometry. Those points were roughly broken as
  - [2 pts] for putting the right limits in the integral (consistent with your choice of geometry);
  - [3 pts] for correctly setting up the integral in space (dx), but not in volume, radius, or any other variable;
- [2 pts] for giving the right expression for the area;
- [2 pts] for the integration/final answer.

Some common errors:

- Heavily flawed answers got [$\leq$ 5 pts].
- Writing an integral in volume instead of length showed little understanding of the physics, so [$\leq$ 6 pts].
- Calculating the resistance directly as $R = \frac{\rho\, l}{A}$, putting some value of A was also heavily flawed yielding [4 pts].

Other points:

- Some students got $\pi\, r^2$ as the expression for the area, but they didn't define r, and their limits of integration was incompatible with r.
- I have seen in some few cases the expression like $\pi\,(R^2 - (l - \frac{R}{2})^2)$ for the area. This expression is perfectly fine if you define $l = 0$ at the left part of the resistor.
- Since $dR = \frac{\rho\, dl}{A}$, you should integrate in dl, or dx, or whichever variable that goes along the slices of your resistor. Some students integrated over $d\theta$. This is wrong: if we have $x = R\cos\theta$, then $dx = -R\sin\theta\, d\theta$.
- I didn't took points of for simplification. Note that $\ln 3 - \ln\left(\frac{1}{3}\right) = \ln 3 + \ln 3 = \ln 9 = 2\ln 3$.