1. Solve linear systems of equations $A\,x = b$, where

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

using the row reduction algorithm.

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 2 & 1 & 0 \\ 1 & 1 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 1 & 2 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\boxed{x = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}}$$

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2. Let

$$A = \begin{array}{c} \begin{matrix} \downarrow & \downarrow & \end{matrix} \\ \begin{pmatrix} 1 & -1 & t \\ 1 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix} \end{array}$$

where t is a real parameter.

(a) For $t = 0$, find a basis of the column space and a basis of the null space of A.

(b) For $t \neq 0$, show that A is invertible.

(a)

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & -2 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{array}{c} \begin{matrix} \downarrow & \downarrow & & \end{matrix} \\ \begin{bmatrix} 1 & 0 & 1/2 & 0 \\ 0 & 1 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$x_1 = -\frac{1}{2}x_3$

$x_2 = -\frac{1}{2}x_3$

x_3 free

Basis for Col $A = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} \right\}$ $\checkmark$ Basis for Nul $A = \left\{ \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix} \right\}$ $\checkmark$

(b) If $t \neq 0$, then $\det A \neq 0$

Proof: $\det A = 0 - 1 - t + 1 - t - 0$

$= -2t$

If $\det A \neq 0$, then A must be invertible.

$$\begin{bmatrix} 1 & -1 & t \\ 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & -1 & t \\ 0 & \boxed{-2} & 1 \\ 0 & 0 & \boxed{t} \end{bmatrix}$$

$\checkmark$

A is has 3 pivots.

A is 3x3 in size.

By IMT, A is invertible, as it has pivots equal to the number of columns.

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3. Let $\mathbb{P}_2$ be the set of polynomials of degree at most 2, and define a map $\mathbf{T}$ from $\mathbb{P}_2$ to $\mathbb{R}$ as follows: let $u(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2$ be a polynomial in $\mathbb{P}_2$. Then $\mathbf{T}(u(x)) = \alpha_0 + \alpha_1 + \alpha_2$.

10 (a) Show that $\mathbf{T}$ is a linear transformation.

7 (b) find the dimensions of the range space and the kernel of $\mathbf{T}$.

(a) $T(u(x))$ is linear if it preserves vector addition and scalar multiplication.
That is: $T(u(x)+v(x)) = T(u(x)) + T(v(x))$ and $T(c\,u(x)) = c\,Tu(x)$.
Let $v(x) = \beta_0 + \beta_1 x + \beta_2 x^2$. $v(x)$ is in $\mathbb{P}_2$. Then $u(x)+v(x) = (\alpha_0+\beta_0)+(\alpha_1+\beta_1)x+(\alpha_2+\beta_2)x^2$

Then $T(u(x)+v(x)) = \alpha_0+\beta_0+(\alpha_1+\beta_1)+(\alpha_2+\beta_2) =$
$= \alpha_0+\alpha_1+\alpha_2) + (\beta_0+\beta_1+\beta_2)$
$= T(u(x)) + T(v(x))$

Let c be a constant scalar value.
Then $c\,u(x) = ca_0 + ca_1 x + ca_2 x^2$
and $T(c\,u(x)) = ca_0 + ca_1 + ca_2$
$= c(a_0+a_1+a_2)$
$= c\,T(u(x))$ Thus, T is a linear transformation.

(b) $\boxed{\dim \text{Range } T = 1}$, as T maps P_2 to R, and R has 1 dimension, so T maps to 1 dimension. onto?

$T(u(x)) = 0 = a_0 + \alpha_1 + \alpha_2 \rightarrow \alpha_0 = -\alpha_1 - \alpha_2$; α_1, α_2 are free,

$\boxed{\dim \text{Ker } T = 2}$, as $\dim \text{Range } T(x) + \dim \ker T(x) = \dim x$;
P_2 is dim 3, dim Range $T = 1$
$3 - 1 = 2 = \dim$ range T.

Also, solution to $T(u(x)) = 0$ can be expressed in terms of 2 free variables, so ker T has dimension 2.

4. Let $\mathbf{V}$ be a vector space, and let $\mathbf{H}$ and $\mathbf{W}$ be two subspaces of $\mathbf{V}$. Define

$$\mathbf{S} = \{u + v | u \in \mathbf{H} \quad \text{and} \quad v \in \mathbf{W}.\}$$

Show that $\mathbf{S}$ is a subspace of $\mathbf{V}$.

To show S is a subspace of V, S must contain $\vec{0}$, and preserve operators vector addition and scalar multiplication.
Let $u \in H$, $v \in W$.
S contains the $\vec{0}$ vector, as $u+v=0$ if both u and v are zero.
u and v can be 0 because $u \in H$, $v \in W$, and H & W are subspaces, so they contain the 0 vector. Thus, S contains the $\vec{0}$ vector.

Let $u,x \in H$ and $v,y \in W$.
Then $u+v$ and $x+y$ must be in S.
If S is a subspace, it should contain $(u+v)+(x+y)$, to preserve vector addition. $u+v+x+y = (u+x)+(v+y)$
$u+x$ is contained in H, as both $u,x \in H$ and H is a subspace
$v+y$ is contained in W, as both $v,y \in W$ and W is a subspace,
Then S must contain $(u+x)+(v+y) = (u+v+x+y)$, as $u+x \in H$ and $v+y \in W$.
Thus, S preserves vector addition.

Let $u \in H$, $v \in Y$, and let c be a constant.
To preserve scalar multiplication, S must contain $c(u+v) = cu+cv$.
cu must be $\in H$, because H is a subspace and $u \in H$. So $cu \in H$.
cv must be $\in W$, because W is a subspace, and $v \in W$. So $cv \in W$.
Then $cu+cv$ must be in S because $cu \in H$, and $cv \in W$.
$cu+cv = c(u+v)$.
Thus S preserves scalar multiplication.

S must be a subspace of V, as it contains $\vec{0}$, and preserves vector addition and scalar multiplication.