CS 70 Second Midterm 7 April 2011

NAME (1 pt): ______________________________

SID (1 pt): ______________________________

TA (1 pt): ______________________________

Name of Neighbor to your left (1 pt): ______________________________

Name of Neighbor to your right (1 pt): ______________________________

Instructions: This is a closed book, closed calculator, closed computer, closed network, open brain exam, but you are permitted a 1 page, double-sided set of notes, large enough to read without a magnifying glass.

You get one point each for filling in the 5 lines at the top of this page. Each other question is worth 20 points.

Write all your answers on this exam. If you need scratch paper, ask for it, write your name on each sheet, and attach it when you turn it in (we have a stapler).

1	
2	
3	
4	
5	
Total	

Answers

Question 1 (20 points). For full credit explain your answers.

Part 1 (10 points). Pick a random integer n in the range from 0 to 9,999,999, each with equal probability. What is the probability that the decimal digits of n add up to 9? Fill in your answer in the box below.

P=

Answer: We need to count the number of ways you can order 9 stars and 6 bars, with the number of stars between bars i and $i+1$ being the value of decimal digit $i+1$: $C(15,6) = 15!/(6!9!)$. (If the sequence starts (ends) with stars, these determine the value of the first (last) decimal digit.) Then we divide by the number of possible 7-digit numbers, 10^7, to get the answer $C(15,6)/10^7$.

Part 2 (5 points). What is the probability that the decimal digits of n add up to 10? Fill in your answer in the box below.

P=

Answer: This is not exactly stars-and-bars with 10 stars and 6 bars, because if all 10 stars end up between two consecutive bars, we can't represent this as a single decimal digit. So we need to subtract out these C(7,1) = 7 possibilities, yielding $(C(16,6) - 7)/10^7$.

Part 3 (5 points). What is the probability that the decimal digits of n add up to 11? Fill in your answer in the box below.

P=

Answer: Again, this is not exactly stars-and-bars with 11 stars and 6 bars, because we need to subtract out the cases where one "digit" (group of consecutive stars) is 10 (and one other is 1) or one "digit" is 11: The number of ways one "digit" could be 10 and another 1 is $7 \cdot 6 = 42$, and the number of ways one "digit" could be 11 is 7, yielding $(C(17,6) - 42 - 7))/10^7$.

Question 1 (20 points). For full credit explain your answers.

Part 1 (10 points). Pick a random integer n in the range from 0 to 99,999,999, each with equal probability. What is the probability that the decimal digits of n add up to 9? Fill in your answer in the box below.

P=

Answer: We need to count the number of ways you can order 9 stars and 7 bars, with the number of stars between bars i and $i+1$ being the value of decimal digit $i+1$: $C(16,7) = 16!/(7!9!)$. (If the sequence starts (ends) with stars, these determine the value of the first (last) decimal digit.) Then we divide by the number of possible 8-digit numbers, 10^8, to get the answer $C(16,7)/10^8$.

Part 2 (5 points). What is the probability that the decimal digits of n add up to 10? Fill in your answer in the box below.

P=

Answer: This is not exactly stars-and-bars with 10 stars and 7 bars, because if all 10 stars end up between two consecutive bars, we can't represent this as a single decimal digit. So we need to subtract out these C(8,1) = 8 possibilities, yielding $(C(17,7) - 8)/10^8$.

Part 3 (5 points). What is the probability that the decimal digits of n add up to 11? Fill in your answer in the box below.

P=

Answer: Again, this is not exactly stars-and-bars with 11 stars and 7 bars, because we need to subtract out the cases where one "digit" (group of consecutive stars) is 10 (and one other is 1) or one "digit" is 11: The number of ways one "digit" could be 10 and another 1 is $8 \cdot 7 = 56$, and the number of ways one "digit" could be 11 is 8, yielding $(C(18,7) - 56 - 8))/10^8$.

Question 2 (20 points). For full credit explain your answers.

1. (*10 points*)

   Let the sample space $\Omega = \{0, 1, 2, 3\}$, and let the probability of each sample point be uniform. What is the probability of the events $A = \{1, 2\}$, $B = \{2, 3\}$, $C = \{1, 3\}$? Are events A and B independent? What is $Pr[A|B \cup C]$?

   Answer:

   $P(A) = P(B) = P(C) = \frac{2}{4}$
   $P(A \cap B) = \frac{1}{4} = P(A)P(B)$, so they are independent
   $Pr[A|B \cup C] = \frac{2}{3}$

2. (*10 points*)

   Suppose you are given a bag containing n unbiased coins. You are told that $n-1$ of these are normal coins, with heads on one side and tails on the other; however, the remaining coin has heads on both its sides.

   - Suppose you reach into the bag, pick out a coin uniformly at random, flip it and get a head. What is the (conditional) probability that this coin you chose is the fake (i.e., double-headed) coin?
     Answer: Let F be the event that the coin we picked from the bag is fake, and N is the event that it is not fake, and let H be the event that the coin we picked from the bag comes up head.
     $P(F|H) = \frac{P(F \cap H)}{P(H)} = \frac{P(F)P(H|F)}{P(H \cap F)+P(H \cap N)} = \frac{P(F)P(H|F)}{P(F)P(H|F)+P(N)P(H|N)} = \frac{\frac{1}{n}}{\frac{1}{n}+\frac{n-1}{n}\times\frac{1}{2}}$

   - Suppose you flip the coin k times after picking it (instead of just once) and see k heads. What is now the conditional probability that you picked the fake coin?
     Answer: Let F be the event that the coin we picked from the bag is fake, and N is the event that it is not fake, and let H^k be the event that the coin we picked from the bag comes up head k times.
     $P(F|H^k) = \frac{P(F \cap H^k)}{P(H^k)} = \frac{P(F)P(H^k|F)}{P(H^k \cap F)+P(H^k \cap N)} = \frac{P(F)P(H^k|F)}{P(F)P(H^k|F)+P(N)P(H^k|N)} = \frac{\frac{1}{n}}{\frac{1}{n}+\frac{n-1}{n}\times\frac{1}{2^k}}$

Question 2 (20 points). For full credit explain your answers.

1. *(10 points)*

   Let the sample space $S = \{a, b, c, d\}$, and let the probability of each sample point be uniform. What is the probability of the events $A = \{b, c\}$, $B = \{c, d\}$, $C = \{b, d\}$? Are events A and B independent? What is $Pr[A|B \cup C]$?

   Answer:

   $P(A) = P(B) = P(C) = \frac{2}{4}$
   $P(A \cap B) = \frac{1}{4} = P(A)P(B)$, so they are independent
   $Pr[A|B \cup C] = \frac{2}{3}$

2. *(10 points)*

   Suppose you are given a bag containing m unbiased coins. You are told that $m - 1$ of these are normal coins, with heads on one side and tails on the other; however, the remaining coin has tails on both its sides.

   - Suppose you reach into the bag, pick out a coin uniformly at random, flip it and get a tail. What is the (conditional) probability that this coin you chose is the fake (i.e., double-tailed) coin?
     Answer: Let F be the event that the coin we picked from the bag is fake, and N is the event that it is not fake, and let T be the event that the coin we picked from the bag comes up tails.
     $P(F|T) = \frac{P(F \cap T)}{P(T)} = \frac{P(F)P(T|F)}{P(T \cap F) + P(T \cap N)} = \frac{P(F)P(T|F)}{P(F)P(T|F) + P(N)P(T|N)} = \frac{\frac{1}{m}}{\frac{1}{m} + \frac{m-1}{m} \times \frac{1}{2}}$

   - Suppose you flip the coin s times after picking it (instead of just once) and see s tails. What is now the conditional probability that you picked the fake coin?
     Answer: Let F be the event that the coin we picked from the bag is fake, and N is the event that it is not fake, and let T^s be the event that the coin we picked from the bag comes up tails s times.
     $P(F|T^s) = \frac{P(F \cap T^s)}{P(T^s)} = \frac{P(F)P(T^s|F)}{P(T^s \cap F) + P(T^s \cap N)} = \frac{P(F)P(T^s|F)}{P(F)P(T^s|F) + P(N)P(T^s|N)} = \frac{\frac{1}{m}}{\frac{1}{m} + \frac{m-1}{m} \times \frac{1}{2^s}}$

Question 3 (20 points) Bayes' Casino.

At Bayes' Casino in Las Vegas, there are two types of slot machines: Red and Blue. Every machine of one color results in a 'win' 10% of the time, and every machine of the other color results in a win 25% of the time. (A 'win' is when the machine returns money). Nobody knows which color wins more frequently, but you are 80% sure it's the Blue machines. You find a Blue machine in the casino and play a quarter.

(**a**) 6 points. Let \$ be the event the Blue machine wins. Let A be the event that the Blue machine is a "good" (25%) one. Write down the following probabilities:

Answer:

$$P(\$|A) = 25\% \qquad P(\overline{\$}|A) = 75\%$$
$$P(\$|\overline{A}) = 10\% \qquad P(\overline{\$}|\overline{A}) = 90\%$$
$$P(A) = 80\% \qquad P(\overline{A}) = 20\%$$

(**b**) 7 points. Suppose the Blue machine does not win. Given this event and your 80% initial estimate, what is the probability that the Blue machines have the better win rate (25%)? Feel free to write your answer as a fraction. Show your work in order to earn any partial credit.

Answer: Using Bayes' rule, then the total probability rule, then Bayes' rule again, we write

$$\begin{aligned} P(A|\overline{\$}) &= \frac{P(A)P(\overline{\$}|A)}{P(\overline{\$})} \\ &= \frac{P(A)P(\overline{\$}|A)}{P(\overline{\$} \cap A) + P(\overline{\$} \cap \overline{A})} \\ &= \frac{P(A)P(\overline{\$}|A)}{P(A)P(\overline{\$}|A) + P(\overline{A})P(\overline{\$}|\overline{A})} \\ &= \frac{30}{39} \approx 77\% \end{aligned}$$

(**c**) 7 points. Repeat part (b), supposing instead that the Blue machine wins.

Answer: Similar to part (b), we have

$$\begin{aligned} P(A|\$) &= \frac{P(A)P(\$|A)}{P(A)P(\$|A) + P(\overline{A})P(\$|\overline{A})} \\ &= \frac{10}{11} \approx 91\% \end{aligned}$$

Question 3 (20 points) Bayes' Casino.

At Bayes' Casino in Las Vegas, there are two types of slot machines: Red and Blue. Every machine of one color results in a 'win' 15% of the time, and every machine of the other color results in a win 20% of the time. (A 'win' is when the machine returns money). Nobody knows which color wins more frequently, but you are 75% sure it's the Blue machines. You find a Blue machine in the casino and play a quarter.

(**a**) 6 points. Let \$ be the event the Blue machine wins. Let A be the event that the Blue machine is a "good" (20%) one. Write down the following probabilities:

Answer:

$$
\begin{aligned}
P(\$|A) &= 20\% & P(\overline{\$}|A) &= 80\% \\
P(\$|\overline{A}) &= 15\% & P(\overline{\$}|\overline{A}) &= 85\% \\
P(A) &= 75\% & P(\overline{A}) &= 25\%
\end{aligned}
$$

(**b**) 7 points. Suppose the Blue machine does not win. Given this event and your 75% initial estimate, what is the probability that the Blue machines have the better win rate (20%)? Feel free to write your answer as a fraction. Show your work in order to earn any partial credit.

Answer: Using Bayes' rule, then the total probability rule, then Bayes' rule again, we write

$$
\begin{aligned}
P(A|\overline{\$}) &= \frac{P(A)P(\overline{\$}|A)}{P(\overline{\$})} \\
&= \frac{P(A)P(\overline{\$}|A)}{P(\overline{\$} \cap A) + P(\overline{\$} \cap \overline{A})} \\
&= \frac{P(A)P(\overline{\$}|A)}{P(A)P(\overline{\$}|A) + P(\overline{A})P(\overline{\$}|\overline{A})} \\
&= \frac{48}{65} \approx 74\%
\end{aligned}
$$

(**c**) 7 points. Repeat part (b), supposing instead that the Blue machine wins.

Answer: Similar to part (b), we have

$$
\begin{aligned}
P(A|\$) &= \frac{P(A)P(\$|A)}{P(A)P(\$|A) + P(\overline{A})P(\$|\overline{A})} \\
&= \frac{4}{5} = 80\%
\end{aligned}
$$

Question 4 (20 points) Binomial Distribution.

4.1 (10 points). Suppose X is a random variable that can take positive integer values $0, 1, \ldots, m$ and β is some real number such that $0 \leq \beta \leq 1$. Distribution of X is given by the following recurrence relation:

$$P(X = k) = \begin{cases} (1-\beta)^m & \text{for } k = 0 \\ \frac{\beta}{1-\beta} \cdot \frac{m-k+1}{k} \cdot P(X = k-1) & \text{for } k = 1, 2, \ldots, m \end{cases}$$

Use induction to prove that X is actually a binomially distributed random variable. What are the parameters of the binomial distribution? What is $E(X)$?

Answer: We claim that $X \sim Bin(m, \beta)$. In order to prove the claim, we have to show that the given recurrence relation can be simplified to the standard form of the binomial distribution. In other words, we have to show that for all $k = 0, 1, \ldots, m$

$$P(X = k) = \binom{m}{k} \beta^k (1-\beta)^{m-k}$$

We will use induction to prove it.

- Base case ($k = 0$):

$$\begin{aligned} P(X = 0) &= \binom{m}{0} \beta^0 (1-\beta)^m \\ &= (1-\beta)^m \end{aligned}$$

- Induction Hypothesis: $P(X = k) = \binom{m}{k} \beta^k (1-\beta)^{m-k}$ for all $0 \leq k \leq t$, and for some t where $0 \leq t < m$. As the induction step, we now have to show that

$$P(X = t+1) = \binom{m}{t+1} \beta^{t+1} (1-\beta)^{m-(t+1)}$$

  Plugging $k = t+1$ in the given recurrence relation, we get

$$\begin{aligned} P(X = t+1) &= \frac{\beta}{1-\beta} \cdot \frac{m-t-1+1}{t+1} \cdot P(X = t) \\ &= \frac{\beta}{1-\beta} \cdot \frac{m-t}{t+1} \cdot \binom{m}{t} \beta^t (1-\beta)^{m-t} \quad \text{[by induction hypothesis]} \\ &= \frac{\beta}{1-\beta} \cdot \frac{m-t}{t+1} \cdot \frac{m!}{t!(m-t)!} \cdot \beta^t (1-\beta)^{m-t} \\ &= \frac{m!}{(t+1)!(m-t-1)!} \cdot \beta^{t+1} (1-\beta)^{m-t-1} \\ &= \binom{m}{t+1} \beta^{t+1} (1-\beta)^{m-(t+1)} \end{aligned}$$

Hence, $X \sim Bin(m, \beta)$, i.e. parameters of the binomial distribution are m and β. Using standard result, we can write $E(X) = m\beta$.

4.2 (10 points). You have two boxes A and B, each containing n balls. You randomly pick one box and then take one ball out of it. You continue this process until you pick a box and find it empty. Suppose X is the number of balls that remain in the other box when you stop. If probability of picking A and B are p and $1-p$ respectively, write down the distribution of X in terms of n and p.

Answer: The event $\{X = k\}$ means that you end up picking an empty box and the other box contains k balls at that iteration. There are two possible cases, *viz.* you pick box A and find it empty while box B contains k balls, or you pick box B and find it empty while box A contains k balls. X is, therefore, an integer-valued random variable that ranges over $\{0, 1, \ldots, n\}$. Let A_k denote the event that you pick box A, find it empty, but still there are k balls in box B. Similarly, let B_k denote the event that you pick box B, find it empty, but still there are k balls in box A. Therefore, we can write

$$P(X = k) = P(A_k) + P(B_k)$$

Now, event A_k only happens after you picked up exactly $n + n - k = 2n - k$ balls, all n balls from box A and $n - k$ balls from box B in some order, and then pick box A again (which is empty by now) at the $(2n-k+1)$-th iteration. Probability of taking out all n balls from box A and $n-k$ balls from box B in $2n-k$ iterations is $\binom{2n-k}{n}p^n(1-p)^{n-k}$ and probability of choosing box A in the $(2n-k+1)$-th iteration is p. Therefore,

$$\begin{aligned} P(A_k) &= \binom{2n-k}{n} p^n (1-p)^{n-k} \cdot p \\ &= \binom{2n-k}{n} p^{n+1} (1-p)^{n-k} \end{aligned}$$

Similarly,

$$\begin{aligned} P(B_k) &= \binom{2n-k}{n} p^{n-k} (1-p)^{n} \cdot (1-p) \\ &= \binom{2n-k}{n} p^{n-k} (1-p)^{n+1} \end{aligned}$$

Hence, the required distribution of X is (for all $k = 0, 1, \ldots, n$),

$$\begin{aligned} P(X = k) &= P(A_k) + P(B_k) \\ &= \binom{2n-k}{n} p^{n+1} (1-p)^{n-k} + \binom{2n-k}{n} p^{n-k} (1-p)^{n+1} \\ &= \binom{2n-k}{n} (p^{n+1} (1-p)^{n-k} + p^{n-k} (1-p)^{n+1}) \end{aligned}$$

Question 4 (20 points) Binomial Distribution.

4.1 (10 points). Suppose X is a random variable that can take positive integer values $0, 1, \ldots, n$ and μ is some real number such that $0 \leq \mu \leq 1$. Distribution of X is given by the following recurrence relation:

$$P(X = k) = \begin{cases} (1-\mu)^n & \text{for } k = 0 \\ \frac{\mu}{1-\mu} \cdot \frac{n-k+1}{k} \cdot P(X = k-1) & \text{for } k = 1, 2, \ldots, n \end{cases}$$

Use induction to prove that X is actually a binomially distributed random variable. What are the parameters of the binomial distribution? What is $E(X)$?

Answer: We claim that $X \sim Bin(n, \mu)$. In order to prove the claim, we have to show that the given recurrence relation can be simplified to the standard form of the binomial distribution. In other words, we have to show that for all $k = 0, 1, \ldots, n$

$$P(X = k) = \binom{n}{k} \mu^k (1-\mu)^{n-k}$$

We will use induction to prove it.

- Base case ($k = 0$):

$$\begin{aligned} P(X = 0) &= \binom{n}{0} \mu^0 (1-\mu)^n \\ &= (1-\mu)^n \end{aligned}$$

- Induction Hypothesis: $P(X = k) = \binom{n}{k} \mu^k (1-\mu)^{n-k}$ for all $0 \leq k \leq t$, and for some t where $0 \leq t < m$. As the induction step, we now have to show that

$$P(X = t+1) = \binom{n}{t+1} \mu^{t+1} (1-\mu)^{n-(t+1)}$$

  Plugging $k = t+1$ in the given recurrence relation, we get

$$\begin{aligned} P(X = t+1) &= \frac{\mu}{1-\mu} \cdot \frac{n-t-1+1}{t+1} \cdot P(X = t) \\ &= \frac{\mu}{1-\mu} \cdot \frac{n-t}{t+1} \cdot \binom{n}{t} \mu^t (1-\mu)^{n-t} \quad \text{[by induction hypothesis]} \\ &= \frac{\mu}{1-\mu} \cdot \frac{n-t}{t+1} \cdot \frac{n!}{t!(n-t)!} \cdot \mu^t (1-\mu)^{n-t} \\ &= \frac{n!}{(t+1)!(n-t-1)!} \cdot \mu^{t+1} (1-\mu)^{n-t-1} \\ &= \binom{n}{t+1} \mu^{t+1} (1-\mu)^{n-(t+1)} \end{aligned}$$

Hence, $X \sim Bin(n, \mu)$, i.e. parameters of the binomial distribution are n and μ. Using standard result, we can write $E(X) = n\mu$.

4.2 (10 points). You have two boxes R and S, each containing m balls. You randomly pick one box and then take one ball out of it. You continue this process until you pick a box and find it empty. Suppose Y is the number of balls that remain in the other box when you stop. If probability of picking R and S are q and $1-q$ respectively, write down the distribution of Y in terms of m and q.

Answer: The event $\{Y = k\}$ means that you end up picking an empty box and the other box contains k balls at that iteration. There are two possible cases, *viz.* you pick box R and find it empty while box S contains k balls, or you pick box S and find it empty while box R contains k balls. Y is, therefore, an integer-valued random variable that ranges over $\{0, 1, \ldots, m\}$. Let R_k denote the event that you pick box R, find it empty, but still there are k balls in box S. Similarly, let S_k denote the event that you pick box S, find it empty, but still there are k balls in box R. Therefore, we can write

$$P(Y = k) = P(R_k) + P(S_k)$$

Now, event R_k only happens after you picked up exactly $m+m-k = 2m-k$ balls, all m balls from box R and $m-k$ balls from box S in some order, and then pick box R again (which is empty by now) at the $(2m-k+1)$-th iteration. Probability of taking out all m balls from box R and $m-k$ balls from box S in $2m-k$ iterations is $\binom{2m-k}{m}q^m(1-q)^{m-k}$ and probability of choosing box R in the $(2m-k+1)$-th iteration is q. Therefore,

$$\begin{aligned} P(R_k) &= \binom{2m-k}{m} q^m (1-q)^{m-k} \cdot q \\ &= \binom{2m-k}{m} q^{m+1} (1-q)^{m-k} \end{aligned}$$

Similarly,

$$\begin{aligned} P(S_k) &= \binom{2m-k}{m} q^{m-k} (1-q)^{m} \cdot (1-q) \\ &= \binom{2m-k}{m} q^{m-k} (1-q)^{m+1} \end{aligned}$$

Hence, the required distribution of Y is (for all $k = 0, 1, \ldots, m$),

$$\begin{aligned} P(Y = k) &= P(R_k) + P(S_k) \\ &= \binom{2m-k}{m} q^{m+1} (1-q)^{m-k} + \binom{2m-k}{m} q^{m-k} (1-q)^{m+1} \\ &= \binom{2m-k}{m} (q^{m+1} (1-q)^{m-k} + q^{m-k} (1-q)^{m+1}) \end{aligned}$$

Question 5 (20 points) Random Variables. Instead of a pair of the usual 6-sided dice, you can play a game with one 4 sided die (sides numbered 1 through 4, each equally likely to come up), and one 8 sided die (sides numbered 1 through 8, again all equally likely). Let A be the sum of the values that come up on these two dice.

5.1 (5 points) What is the expected value of A?

$E(A) =$

Answer:

$$\begin{aligned}
E(A) &= E(D4) + E(D8) \\
E(D4) &= \sum_{i=1}^{4} i * \frac{1}{4} \\
&= \frac{1}{4}\sum_{i=1}^{4} i \\
&= \frac{10}{4} \\
&= 2.5 \\
E(D8) &= \sum_{i=1}^{8} i * \frac{1}{8} \\
&= \frac{1}{8}\sum_{i=1}^{8} \\
&= \frac{36}{8} \\
&= 4.5 \\
E(A) &= 2.5 + 4.5 \\
E(A) &= 7
\end{aligned}$$

5.2 (5 points) What is the probability $A = 8$?

$P(A = 8) =$

Answer: $A = 8$ when we roll either (1,7), (2,6), (3,5), or (4,4)

$$\begin{aligned}
P(A = 8) &= 4 * P(D4 = x) * P(D8 = y) \\
&= 4 * \frac{1}{4} * \frac{1}{8} \\
&= \frac{4}{32} \\
&= \frac{1}{8}
\end{aligned}$$

5.3 (10 points) You play a friendly betting game with your friend where you roll two dice each round and if the sum of the two dice is 7 your friend pays you \$5, otherwise you pay your friend \$1. If you play this game with two 6-sided dice your expected profit is \$0 each round. Define a random variable and use it to calculate the expected amount of money you win or lose in 1 round if you play using a 4-sided die and an 8-sided die. In the box below specify whether you win or lose money (by circling the appropriate word) and fill in the expected amount of money you win or lose in one round.

In one round you expect to **win/lose (circle one)** \$

Answer: Let W be the amount of money we win. If we roll a 7 then $W = 5$, otherwise $W = -1$. There are 4 events where $A = 7$, we roll either (1,6), (2,5), (3,4), or (4,3). The probability $A = 7$ is $\frac{4}{36} = \frac{1}{8}$. The probability A is any other number is $\frac{7}{8}$.

$$\begin{aligned} E(W) &= 5 * P(A = 7) - 1 * P(A! = 7) \\ &= 5 * \frac{1}{8} - 1 * \frac{7}{8} \\ &= \frac{5}{8} - 1 * \frac{7}{8} \\ &= -\frac{2}{8} \\ &= -\frac{1}{4} \end{aligned}$$

We expect to lose a quarter each round.

Note: For a pair of 6-sided dice $P(A = 7) = \frac{1}{6}$

$$\begin{aligned} E(W) &= 5 * P(A = 7) - 1 * P(A! = 7) \\ &= 5 * \frac{1}{6} - 1 * \frac{5}{6} \\ &= \frac{5}{6} - 1 * \frac{5}{6} \\ &= 0 \end{aligned}$$

Question 5 (20 points) Random Variables. Instead of a pair of the usual 6-sided dice, you can play a game with one 4 sided die (sides numbered 1 through 4, each equally likely to come up), and one 8 sided die (sides numbered 1 through 8, again all equally likely). Let S be the sum of the values that come up on these two dice.

5.1 (5 points) What is the expected value of S?

$E(S) =$

Answer:

$$\begin{aligned} E(S) &= E(D4) + E(D8) \\ E(D4) &= \sum_{i=1}^{4} i * \frac{1}{4} \\ &= \frac{1}{4}\sum_{i=1}^{4} i \\ &= \frac{10}{4} \\ &= 2.5 \\ E(D8) &= \sum_{i=1}^{8} i * \frac{1}{8} \\ &= \frac{1}{8}\sum_{i=1}^{8} \\ &= \frac{36}{8} \\ &= 4.5 \\ E(S) &= 2.5 + 4.5 \\ E(S) &= 7 \end{aligned}$$

5.2 (5 points) What is the probability $S = 8$?

$P(S = 8) =$

Answer: $S = 8$ when we roll either (1,7), (2,6), (3,5), or (4,4)

$$\begin{aligned} P(S = 8) &= 4 * P(D4 = x) * P(D8 = y) \\ &= 4 * \frac{1}{4} * \frac{1}{8} \\ &= \frac{4}{32} \\ &= \frac{1}{8} \end{aligned}$$

5.3 (10 points) You play a friendly betting game with your friend where you roll two dice each round and if the sum of the two dice is 7 your friend pays you \$5, otherwise you pay your friend \$1. If you play this game with two 6-sided dice your expected profit is \$0 each round. Define a random variable and use it to calculate the expected amount of money you win or lose in 1 round if you play using a 4-sided die and an 8-sided die. In the box below specify whether you win or lose money (by circling the appropriate word) and fill in the expected amount of money you win or lose in one round.

In one round you expect to **win/lose (circle one)** \$

Answer: Let W be the amount of money we win. If we roll a 7 then $W = 5$, otherwise $W = -1$. There are 4 events where $S = 7$, we roll either (1,6), (2,5), (3,4), or (4,3). The probability $S = 7$ is $\frac{4}{36} = \frac{1}{8}$. The probability S is any other number is $\frac{7}{8}$.

$$\begin{aligned} E(W) &= 5 * P(S = 7) - 1 * P(S! = 7) \\ &= 5 * \frac{1}{8} - 1 * \frac{7}{8} \\ &= \frac{5}{8} - 1 * \frac{7}{8} \\ &= -\frac{2}{8} \\ &= -\frac{1}{4} \end{aligned}$$

We expect to lose a quarter each round.

Note: For a pair of 6-sided dice $P(S = 7) = \frac{1}{6}$

$$\begin{aligned} E(W) &= 5 * P(S = 7) - 1 * P(S! = 7) \\ &= 5 * \frac{1}{6} - 1 * \frac{5}{6} \\ &= \frac{5}{6} - 1 * \frac{5}{6} \\ &= 0 \end{aligned}$$