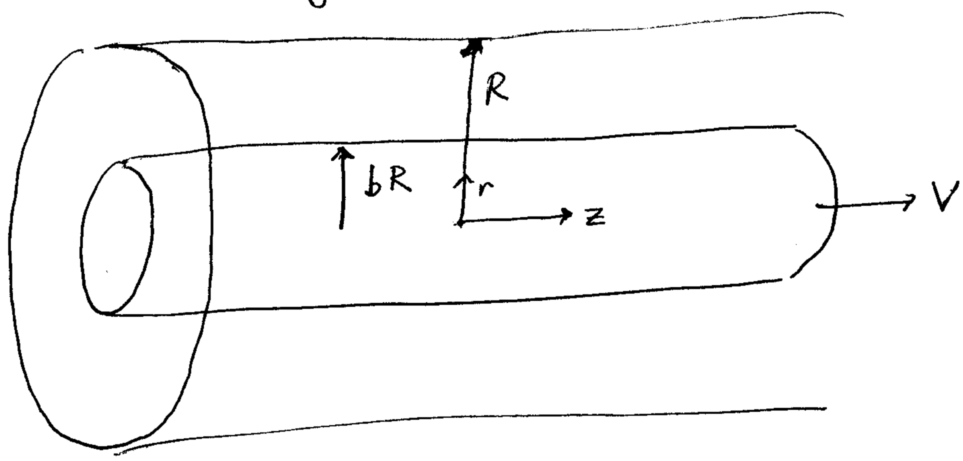
1. For flow in the annular region



Assume : steady-state
incompressible , POWER-LAW FLUID
$L >> R$, $L >> bR$ $\Rightarrow$ neglect entrance & exit effects
assume fully-developed over region away from ends
assume flow is only in z-direction (no swirling flow, $v_r = 0$)
Axisymmetric

$$\Rightarrow \quad v_r = 0, \quad v_\theta = 0, \quad v_z = v_z(r)$$

since there is no pressure gradient in z-direction, $\frac{\partial P}{\partial z} = 0$

with these assumptions, given power-law fluid, we have

$$\eta = K \left| \tfrac{1}{2} \text{II} \right|^{(n-1)/2}$$

From Table 8-1 (cylindrical coords):

$$\tfrac{1}{2}\text{II} = \left(\frac{dv_z}{dr}\right)^2$$

$$\eta = K \left| \left(\frac{dv_z}{dr}\right)^2 \right|^{\left(\frac{n-1}{2}\right)}$$

Since v_z will be maximum at $r = bR$ & decrease with increasing r, $\frac{dv_z}{dr} < 0$

since $\frac{dv_z}{dr} < 0$, this is

$$\eta = K \left| \left(\frac{dv_z}{dr}\right)^2 \right|^{\left(\frac{n-1}{2}\right)} = K \left(-\frac{dv_z}{dr}\right)^{n-1}$$

Looking at Table 7-6, for power-law fluid, τ_{ij} is given by substituting expression above for η into Table 7-6 expressions.

Based on form of $\underline{v}$ assumed, the only non-zero component of stress is

$$\tau_{rz} = \tau_{zr} = \eta \frac{dv_z}{dr} = K\left(-\frac{dv_z}{dr}\right)^{n-1} \frac{dv_z}{dr}$$

Before proceeding, check continuity:

$$\cancelto{0}{\frac{\partial \rho}{\partial t}} = -\frac{1}{r}\frac{\partial}{\partial r}\cancelto{0}{(\rho r v_r)} + \frac{1}{r}\frac{\partial}{\partial \theta}\cancelto{0}{(\rho v_\theta)} + \cancelto{0}{\frac{\partial}{\partial z}}(\rho v_z) \quad \checkmark$$

Assumed form of $\underline{v}$ is consistent w/ continuity.

Cauchy momentum equations (NOT Navier-Stokes since non-Newtonian):

r-comp: $\quad 0 = -\frac{\partial P}{\partial r} \quad \Rightarrow \quad P \neq P(r)$

θ-comp: $\quad 0 = -\frac{1}{r}\frac{\partial P}{\partial \theta} \quad \Rightarrow \quad P \neq P(\theta)$

& $\quad P \neq P(z) \quad \Rightarrow \quad P = \text{constant}$

z-comp:

$$0 = \frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz})$$

$$r\tau_{rz} = C_1 \quad \Rightarrow \quad \tau_{rz} = \frac{C_1}{r}$$

From constitutive equation:

$$K\left(-\frac{dv_z}{dr}\right)^{n-1}\frac{dv_z}{dr} = \frac{C_1}{r}$$

$$\left(-\frac{dv_z}{dr}\right)^{n} = -\frac{C_1}{K}\frac{1}{r}$$

$$-\frac{dv_z}{dr} = \left(-\frac{C_1}{K}\right)^{\frac{1}{n}}\left(\frac{1}{r}\right)^{1/n} = \left(-\frac{C_1}{K}\right)^{\frac{1}{n}} r^{-\frac{1}{n}}$$

$$\frac{dv_z}{dr} = -\left(\frac{-C_1}{K}\right)^{\frac{1}{n}} r^{-\frac{1}{n}}$$

$$v_z = -\left(-\frac{C_1}{K}\right)^{\frac{1}{n}}\left(\frac{n}{n-1}\right) r^{1-\frac{1}{n}} + C_2$$

BC's : at $r = bR$ $\quad v_z = V$

at $r = R$ $\quad v_z = 0$

Apply 2nd BC :

$$0 = -\left(-\frac{C_1}{K}\right)^{\frac{1}{n}}\left(\frac{n}{n-1}\right) R^{1-\frac{1}{n}} + C_2$$

$$C_2 = \left(-\frac{C_1}{K}\right)^{\frac{1}{n}}\left(\frac{n}{n-1}\right) R^{1-\frac{1}{n}}$$

$$v_z = \left(-\frac{C_1}{K}\right)^{\frac{1}{n}}\left(\frac{n}{n-1}\right) R^{1-\frac{1}{n}}\left[1 - \left(\frac{r}{R}\right)^{1-\frac{1}{n}}\right]$$

Applying 1st BC :

$$V = \left(-\frac{C_1}{K}\right)^{\frac{1}{n}}\left(\frac{n}{n-1}\right) R^{1-\frac{1}{n}}\left[1 - b^{1-\frac{1}{n}}\right] \qquad (*)$$

$$\Rightarrow \quad \frac{v_z}{V} = \frac{1 - \left(\frac{r}{R}\right)^{1-\frac{1}{n}}}{1 - b^{1-\frac{1}{n}}}$$

or

$$\boxed{v_z = V \, \frac{1 - \left(\frac{r}{R}\right)^{1-\frac{1}{n}}}{1 - b^{1-\frac{1}{n}}}}$$

Note that if you did the Newtonian fluid case, you would find

$$v_z = V \, \frac{\ln\left(\frac{r}{R}\right)}{\ln(b)}$$

this result can be recovered from the power-law result above, taking the limit $n \to 1$ ($\frac{0}{0}$ using L'Hopital's rule)

Calculate the drag on the rod:

$$F_{drag} = \int_{\theta=0}^{2\pi} \int_{z=0}^{z=L} \tau_{rz}\Big|_{r=bR} \; bR \, d\theta \, dz$$

we can determine $\tau_{rz}\Big|_{r=bR}$ either by noting that

$\tau_{rz} = \dfrac{C_1}{r}$ and solving for the value of C_1 from above (*)

$$\left(\frac{-C_1}{K}\right)^{\frac{1}{n}} = \frac{V\left(1-\frac{1}{n}\right)}{R^{1-\frac{1}{n}}\left[1-b^{1-\frac{1}{n}}\right]}$$

$$C_1 = -K\left[\frac{V\left(1-\frac{1}{n}\right)}{R^{1-n}\left[1-b^{1-\frac{1}{n}}\right]}\right]^n$$

OR by differentiating v_z & using constitutive equation for τ_{rz} in terms of $\frac{dv_z}{dr}$

$$F_{drag} = \int_{\theta=0}^{2\pi} \int_{z=0}^{L} \frac{C_1}{bR} \, bR \, d\theta \, dz$$

$$\boxed{F_{drag} = 2\pi L C_1 = -2\pi L K \left[\frac{V\left(1-\frac{1}{n}\right)}{R^{1-n}\left(1-b^{1-\frac{1}{n}}\right)}\right]^n}$$

The equation to calculate the volumetric flow rate is

$$\boxed{Q = \int_{\theta=0}^{2\pi} \int_{r=bR}^{r=R} v_z \, r \, dr \, d\theta = 2\pi \int_{r=bR}^{R} V \frac{\left(1-\left(\frac{r}{R}\right)^{1-\frac{1}{n}}\right)}{\left(1-b^{1-\frac{1}{n}}\right)} r \, dr}$$

2. Assume incompressible fluid
steady - state
Newtonian
Symmetry with respect to ϕ

$V_r = 0$

$V_\theta = 0$

$V_\phi = V_\phi(r, \theta)$ only (not a function of ϕ due to axisymmetry).

Creeping Flow

Continuity :

$$0 = 0 + 0 + 0 \quad ✓$$

<u>B C's</u> at $r \to \infty$ $\quad V_r = V_\theta = V_\phi = 0$

at $r = R$ $\quad v_r = V_\theta = 0, \quad V_\phi = (R \sin\theta)\,\omega$

$\Rightarrow$ Assume $V_\phi = f(r) \sin\theta$ everywhere

<u>Navier-Stokes</u> (Table 7-10)

r-comp. $\quad 0 = -\dfrac{\partial P}{\partial r} \quad \Rightarrow P \neq P(r)$

creeping flow

θ-comp $\quad 0 = -\dfrac{1}{r}\dfrac{\partial P}{\partial \theta} \quad \Rightarrow P \neq P(\theta)$

ϕ-comp $\quad 0 = \eta\left[\dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2 \dfrac{\partial v_\phi}{\partial r}\right) + \dfrac{1}{r^2 \sin\theta}\dfrac{\partial}{\partial \theta}\left(\sin\theta \dfrac{\partial v_\phi}{\partial \theta}\right) - \dfrac{v_\phi}{r^2 \sin^2\theta}\right]$

Substit. in $v_\phi = f \sin\theta$ into ϕ-comp :

$$0 = \sin\theta \frac{d}{dr}\left(r^2 \frac{df}{dr}\right) + \frac{f}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\cos\theta\right) - \frac{f\sin\theta}{\sin^2\theta}$$

$$0 = \sin\theta \frac{d}{dr}\left(r^2 \frac{df}{dr}\right) + \frac{f}{\sin\theta}\left(\cos^2\theta - \sin^2\theta\right) - \frac{f}{\sin\theta}$$

$$= \sin\theta \frac{d}{dr}\left(r^2 \frac{df}{dr}\right) + \frac{f}{\sin\theta}\left(\cos^2\theta - \sin^2\theta - 1\right)$$

$\cos^2\theta = 1 - \sin^2\theta$

$$= \sin\theta \frac{d}{dr}\left(r^2 \frac{df}{dr}\right) + \frac{f}{\sin\theta}\left(-2\sin^2\theta\right)$$

$$\boxed{0 = \frac{d}{dr}\left(r^2 \frac{df}{dr}\right) - 2f}$$

Solution will be of form $f = r^n$, substituting this in:

$$\frac{d}{dr}\left(r^2 n r^{n-1}\right) - 2r^n = 0$$

$$n\frac{d}{dr}\left(r^{n+1}\right) - 2r^n = 0$$

$$n(n+1)r^n - 2r^n = 0$$

$$n(n+1) - 2 = 0$$

$$n^2 + n - 2 = (n+2)(n-1) = 0$$

$$\Rightarrow \quad n = -2, +1$$

$$f = \frac{C_1}{r^2} + C_2 r$$

as $r \to \infty$, $V_\phi = 0$, so $C_2 = 0$

at $r = R$, $f = R\omega = \frac{C_1}{R^2}$ $\Rightarrow$ $C_1 = R^3\omega$

$$\Rightarrow \quad \boxed{v_\phi = \frac{R^3\omega}{r^2}\sin\theta}$$

3. Assuming boundary layers remain laminar, for these flat plates, we can use the result

$$\tau_w = 0.332\, \eta U \left(\frac{\rho U}{\eta x}\right)^{1/2}$$

$$F_{Drag}^{A} = \int_{z=0}^{z=L} \int_{x=0}^{x=L} 0.332\, \eta U \left(\frac{\rho U}{\eta x}\right)^{1/2} dx\, dz$$

$$= \underbrace{0.332\, \eta U \left(\frac{\rho U}{\eta}\right)^{\frac{1}{2}}}_{C_1} \int_{z=0}^{z=L} \int_{x=0}^{x=L} x^{-1/2}\, dx\, dz$$

$$F_{Drag}^{A} = C_1 L \left(2x^{1/2}\right)\Big/_0^L = C_1 L \left(2L^{1/2} - 0\right)$$

$$= 2 C_1 L^{3/2}$$

$$F_{Drag}^{B} = C_1 \int_{z=0}^{z=L/4} \int_{x=0}^{x=4L} x^{-1/2}\, dx\, dz$$

$$= C_1 \left(\frac{L}{4}\right)\left(2\left(2L^{1/2}\right)\right) = C_1 L^{3/2}$$

$$F_{Drag}^{C} = C_1 \int_{z=0}^{z=4L} \int_{x=0}^{x=L/4} x^{-1/2}\, dx\, dz$$

$$= C_1 (4L)\left(2\left(\frac{L^{1/2}}{2}\right)\right) = 4 C_1 L^{3/2}$$

So $F_{Drag}^{A} : F_{Drag}^{B} : F_{Drag}^{C}$ is $2 : 1 : 4$

b). The drag is the integral of the shear stress over the plate.

The shear stress is proportional to the velocity gradient at the plate surface.

Near the leading edge, the velocity gradients are very large since the boundary layer is very thin.

Thus, C has highest drag, A has next highest, & B the lowest since in B, the velocity gradients diminish along the x-direction.