First Midterm Examination
Closed Books and Closed Notes

Question 1
A Planar Pendulum (25 Points)

As shown in Figure 1, a particle of mass m is attached to a fixed point O by a linearly elastic spring. The spring has a stiffness K and an unstretched length L. The motion of the particle is on the $\mathbf{E}_x - \mathbf{E}_y$ plane.

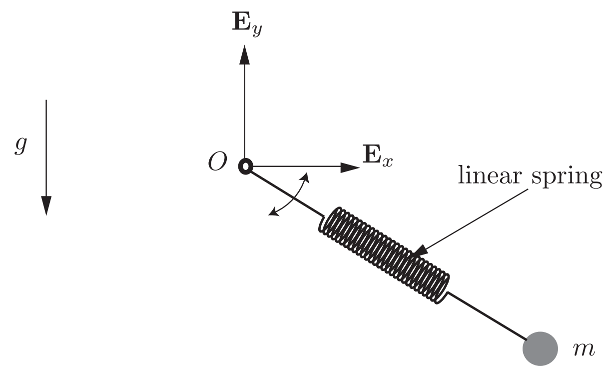


Figure 1: *Schematic of a particle of mass m which is attached to a fixed point O by a linearly elastic spring. A vertical gravitational force $-mg\mathbf{E}_y$ acts on the particle.*

(a) Starting from the standard representations for the position vector

$$\mathbf{r} = x\mathbf{E}_x + y\mathbf{E}_y = r\mathbf{e}_r, \tag{1}$$

establish expressions for the velocity $\mathbf{v}$ and acceleration $\mathbf{a}$ vectors of the particle. In your solution, it is not necessary to derive the intermediate results $\dot{\mathbf{e}}_r = \dot{\theta}\mathbf{e}_\theta$ and $\dot{\mathbf{e}}_\theta = -\dot{\theta}\mathbf{e}_r$.

(b) Draw a freebody diagram of the particle. Your freebody diagram should include a normal force $N\mathbf{E}_z$ and a clear expression for the spring force.

(c) Show that the differential equations governing the motion of the particle are

$$m\left(\ddot{r} - r\dot{\theta}^2\right) = -K\left(r - L\right) - mg\sin(\theta), \qquad m\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right) = -mg\cos(\theta). \tag{2}$$

What is the normal force acting on the particle?

(d) Show that the differential equations governing the motion of the particle can also be expressed in the form

$$\begin{aligned} m\ddot{x} &= -K\left(\sqrt{x^2+y^2} - L\right)\frac{x}{\sqrt{x^2+y^2}}, \\ m\ddot{y} &= -mg - K\left(\sqrt{x^2+y^2} - L\right)\frac{y}{\sqrt{x^2+y^2}}. \end{aligned} \tag{3}$$

(e) With the help of (2), show that it is possible for the particle to be at rest with $\theta = 270^\circ$ and $r = \frac{mg}{K} + L$. Give a physical interpretation of this result.

Question 2

A Particle on a Helix (25 Points)

As shown in Figure 2, a bead of mass m is free to move on a rough curve in the shape of a right-handed circular helix. In addition to friction and normal forces, a vertical gravitational force $-mg\mathbf{E}_z$ acts on the bead.

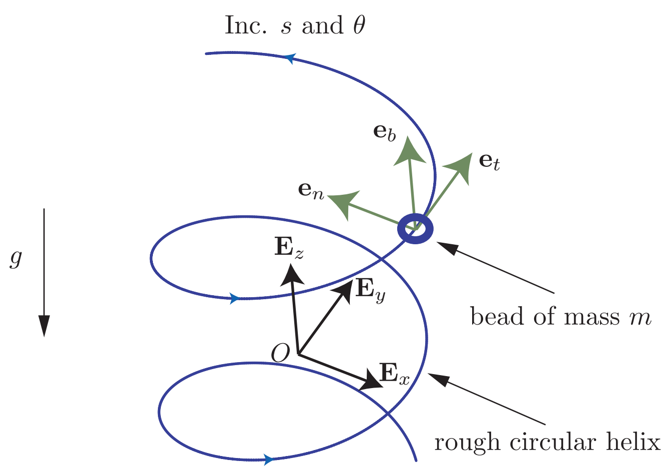


Figure 2: *A particle of mass m moving on a rough circular helix.*

(a) Using a cylindrical polar coordinate system, the position vector of a particle moving on the helix can be described as

$$\mathbf{r} = R\mathbf{e}_r + \alpha R\theta\mathbf{E}_z. \tag{4}$$

Derive expressions for the speed v, and velocity vector $\mathbf{v}$ and acceleration vector $\mathbf{a}$ vector of the particle.

(b) From your results in **(a)** and assuming that $\dot{\theta} > 0$, show that the Frenet triad for the helix is

$$\mathbf{e}_t = \frac{1}{\sqrt{1+\alpha^2}}\left(\mathbf{e}_\theta + \alpha\mathbf{E}_z\right), \qquad \mathbf{e}_n = -\mathbf{e}_r, \qquad \mathbf{e}_b = \frac{1}{\sqrt{1+\alpha^2}}\left(-\alpha\mathbf{e}_\theta + \mathbf{E}_z\right), \tag{5}$$

What is the curvature κ of the helix?

(c) Draw a freebody diagram of the particle. Give clear expressions for the forces acting on the particle, and distinguish the static friction and dynamic friction cases.

(d) Suppose that the particle is moving on the curve with $\dot{\theta} > 0$. Show that the equation governing the motion of the particle is

$$mR\sqrt{1+\alpha^2}\ddot{\theta} = -\frac{mg\alpha}{\sqrt{1+\alpha^2}} - \mu_d\left\|\mathbf{N}\right\|, \tag{6}$$

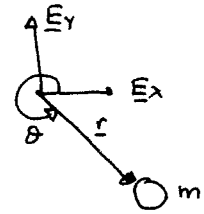
where $\mathbf{N}$ is the normal force. How would you determine $\mathbf{N}$?

(e) Suppose that the particle is stationary at a point on the helix. Show that the friction force and normal force acting on the particle are

$$\mathbf{F}_f = \frac{mg\alpha}{\sqrt{1+\alpha^2}}\mathbf{e}_t, \qquad \mathbf{N} = \frac{mg}{\sqrt{1+\alpha^2}}\mathbf{e}_b. \tag{7}$$

Show that the particle will remain stationary provided $\alpha \leq \mu_s$. Give a physical interpretation of this result.

QUESTION 1



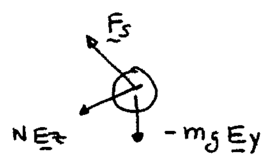
(a) $\underset{\sim}{r} = x\underset{\sim}{E}_x + y\underset{\sim}{E}_y = r\underset{\sim}{e}_r$

$\underset{\sim}{v} = \dot{x}\underset{\sim}{E}_x + \dot{y}\underset{\sim}{E}_y = \dot{r}\underset{\sim}{e}_r + r\dot{\underset{\sim}{e}}_r = \dot{r}\underset{\sim}{e}_r + r\dot{\theta}\underset{\sim}{e}_\theta$

$\underset{\sim}{a} = \ddot{x}\underset{\sim}{E}_x + \ddot{y}\underset{\sim}{E}_y = \ddot{r}\underset{\sim}{e}_r + \dot{r}\dot{\theta}\underset{\sim}{e}_\theta + \dot{r}\dot{\theta}\underset{\sim}{e}_\theta + r\ddot{\theta}\underset{\sim}{e}_\theta + r\dot{\theta}\,(\dot{\underset{\sim}{e}}_\theta = -\dot{\theta}\underset{\sim}{e}_r)$

$= \ddot{x}\underset{\sim}{E}_x + \ddot{y}\underset{\sim}{E}_y = (\ddot{r} - r\dot{\theta}^2)\underset{\sim}{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\underset{\sim}{e}_\theta$

(b)



$\underset{\sim}{F}_s = -K(\|\underset{\sim}{r}\| - L)\dfrac{\underset{\sim}{r}}{\|\underset{\sim}{r}\|} = -K(r-L)\underset{\sim}{e}_r$

$\|\underset{\sim}{r}\| = r = \sqrt{x^2+y^2}$

$\underset{\sim}{F} = -mg\underset{\sim}{E}_y + N\underset{\sim}{E}_z + \underset{\sim}{F}_s$, $-mg\underset{\sim}{E}_Y = -mg\cos\theta\,\underset{\sim}{e}_\theta - mg\sin\theta\,\underset{\sim}{e}_r$

(c) $\underset{\sim}{F} = m\underset{\sim}{a}$:

$\cdot\underset{\sim}{e}_r$: $m(\ddot{r} - r\dot{\theta}^2) = -mg\underset{\sim}{E}_y\cdot\underset{\sim}{e}_r + \underset{\sim}{F}_s\cdot\underset{\sim}{e}_r + N\underset{\sim}{E}_z\cdot\underset{\sim}{e}_r$
$= -mg\sin\theta - K(r-L)$

$\cdot\underset{\sim}{e}_\theta$: $m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = -mg\underset{\sim}{E}_y\cdot\underset{\sim}{e}_\theta + \underset{\sim}{F}_s\cdot\underset{\sim}{e}_\theta + N\underset{\sim}{E}_z\cdot\underset{\sim}{e}_\theta$

$= -mg\cos\theta$

~~(d)~~ ~~E~~ $\cdot\underset{\sim}{E}_z$ $0 = N$

Hence $\underset{\sim}{N} = N\underset{\sim}{E}_z = 0$

(d) $\underset{\sim}{F} = m\underset{\sim}{a}$

$\underset{\sim}{F}_s = -K(\|\underset{\sim}{r}\| - L)\dfrac{x\underset{\sim}{E}_x + y\underset{\sim}{E}_y}{r}$
where $r = \sqrt{x^2+y^2} = \|\underset{\sim}{r}\|$

$\cdot\underset{\sim}{E}_x$ $m\ddot{x} = -mg\underset{\sim}{E}_y\cdot\underset{\sim}{E}_x + N\underset{\sim}{E}_z\cdot\underset{\sim}{E}_x + \underset{\sim}{F}_s\cdot\underset{\sim}{E}_x$

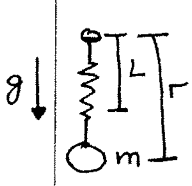
$= -K\left(\sqrt{x^2+y^2} - L\right)\dfrac{x}{\sqrt{x^2+y^2}}$

$\cdot\underset{\sim}{E}_y$ $m\ddot{y} = -mg\underset{\sim}{E}_y\cdot\underset{\sim}{E}_y + N\underset{\sim}{E}_z\cdot\underset{\sim}{E}_y + \underset{\sim}{F}_s\cdot\underset{\sim}{E}_y$

$= -mg - K\left(\sqrt{x^2+y^2} - L\right)\dfrac{y}{\sqrt{x^2+y^2}}$

(e) If particle is at rest $\ddot{x} = \ddot{y} = 0$, $x = r\cos\theta = r\cos(270°) = 0$
$y = r\sin\theta = -r$

Hence $0 = -K\left(\sqrt{x^2+y^2} - L\right)\dfrac{x}{\sqrt{x^2+y^2}} \Rightarrow x = 0$ or $\sqrt{x^2+y^2} = L$



$0 = -mg - K\left(\sqrt{x^2+y^2} - L\right)\dfrac{y}{\sqrt{x^2+y^2}} \Rightarrow$ setting $x = 0$ we find that

$-mg + K(r-L) = 0 \Rightarrow r = \dfrac{mg}{K} + L$. as was to be shown.

The particle hangs vertically downwards and the spring supports its weight.

Alternatively, Consider (2) and set $\theta = 270°$, $r = \frac{mg}{K} + L$, $\dot{\theta} = 0$, $\dot{r} = 0$

$$\Rightarrow \quad m(\ddot{r} - r\dot{\theta}^2) = -K(r-L) - mg\sin\theta$$

simplifies to.
$$m\ddot{r} = -K\left(\frac{mg}{K} + L - L\right) - mg\sin(270°) \Rightarrow$$
$$= -mg - mg(-1)$$
$$= 0$$

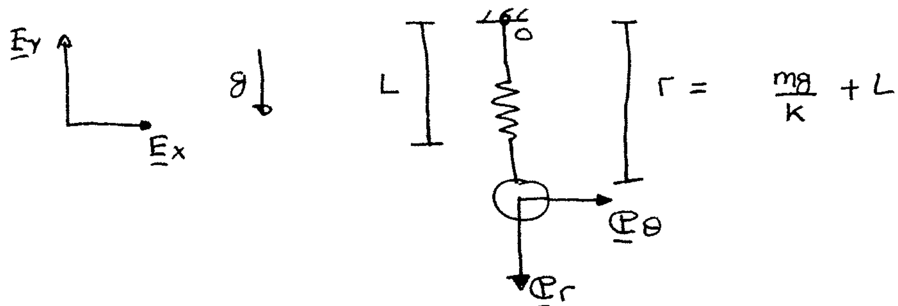
and
$$m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = -mg\cos\theta$$

simplifies to
$$mr\ddot{\theta} = -mg\cos(270°) = -mg(0)$$

Hence $\ddot{\theta} = \ddot{r} = 0 \quad \Rightarrow \quad \dot{\theta}$ and $\dot{r}$ remain 0

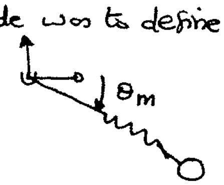
$\Rightarrow \quad \theta$ stays at $270°$ and r stays at $\frac{mg}{K} + L$

Thus it is possible for the ~~spr~~ mass to hang vertically downwards with the spring force balancing gravity



$\boxed{\text{Common Errors.}}$

1. one error that several students made was to define θ as θ_m



This is incorrect, and leads to problems with e_θ and e_r

2. Another common error was to give an incorrect prescription for $\underset{\sim}{F_s}$.

QUESTION 2

(a) $\underline{r} = R\underline{e}_r + R\alpha\theta\underline{E}_z$

$\underline{v} = R\dot{\underline{e}}_r + R\alpha\dot{\theta}\underline{E}_z = R\dot{\theta}(\underline{e}_\theta + \alpha\underline{E}_z)$

$\underline{a} = R\ddot{\theta}(\underline{e}_\theta + \alpha\underline{E}_z) - R\dot{\theta}^2\underline{e}_r = \dot{v}\underline{e}_t + \kappa v^2\underline{e}_n$

$v = \|\underline{v}\| = R\dot{\theta}\sqrt{1+\alpha^2}$ $\quad(\dot{\theta} > 0$ assumed$)$

(b) $\underline{e}_t = \dfrac{\underline{v}}{v} = \dfrac{1}{\sqrt{1+\alpha^2}}(\underline{e}_\theta + \alpha\underline{E}_z)$

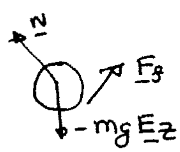
Now $\underline{a} = \dot{v}\underline{e}_t + \kappa v^2\underline{e}_n$. Hence $-R\dot{\theta}^2\underline{e}_r = \kappa v^2\underline{e}_n$

$\Rightarrow \kappa\underline{e}_n = -\dfrac{R\dot{\theta}^2}{v^2}\underline{e}_r = \dfrac{-R\dot{\theta}^2}{R^2(1+\alpha^2)\dot{\theta}^2}\underline{e}_r$

$\Rightarrow \underline{e}_n = -\underline{e}_r$ and $\kappa = \dfrac{1}{R(1+\alpha^2)}$

$\underline{e}_b = \underline{e}_t \times \underline{e}_n = \dfrac{1}{\sqrt{1+\alpha^2}}(\underline{e}_\theta + \alpha\underline{E}_z)\times(-\underline{e}_r) = \dfrac{1}{\sqrt{1+\alpha^2}}\left(\underline{E}_z - \alpha\underline{e}_\theta\right)$

(c)



$\underline{N} = N_n\underline{e}_n + N_b\underline{e}_b$

$\underline{F}_f = F_f\,\underline{e}_t$ (static)

$= -\mu_K\|\underline{N}\|\underline{e}_t$ (dynamic) $\quad \dfrac{\underline{v}}{\|\underline{v}\|} = \underline{e}_t$

(d) $\underline{F} = m\underline{a}$: $\cdot\,\underline{e}_t$ $\quad m\dot{v} = -mg\underline{E}_z\cdot\underline{e}_t + \underline{F}_f\cdot\underline{e}_t + \underline{N}\cdot\underline{e}_t$

$\Rightarrow mR\ddot{\theta}\sqrt{1+\alpha^2} = -\dfrac{mg\alpha}{\sqrt{1+\alpha^2}} - \mu_K\|\underline{N}\|$

$\underline{N}$ can be found from $(\underline{F} = m\underline{a})\cdot\underline{e}_n$ and $(\underline{F} = m\underline{a})\cdot\underline{e}_b$

(e) As particle in stationary friction is static and $\underline{a} = 0$

$\underline{F}_f + \underline{N} - mg\underline{E}_z = \underline{0}$

$\cdot\,\underline{e}_t \quad F_f - \dfrac{mg\alpha}{\sqrt{1+\alpha^2}} = 0$; $\quad \cdot\,\underline{e}_n \quad N_n = mg\underline{E}_z\cdot(-\underline{e}_r) = 0$

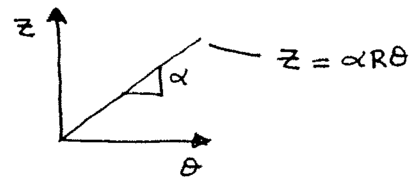
$\cdot\,\underline{e}_b \quad N_b = mg\underline{E}_z\cdot\underline{e}_b = \dfrac{mg}{\sqrt{1+\alpha^2}}$;

Hence $\underline{F}_f = \dfrac{mg\alpha}{\sqrt{1+\alpha^2}}\underline{e}_t$

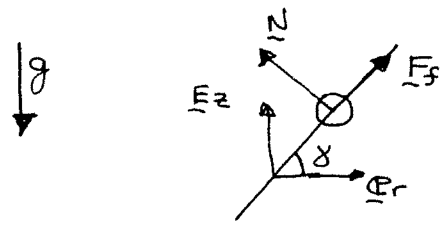
$\underline{N} = \dfrac{mg}{\sqrt{1+\alpha^2}}\underline{e}_b$

From static friction criterion $\|\underline{F}_f\| \le \mu_s\|\underline{N}\| \Rightarrow \alpha \le \mu_s$

Physically α represents the steepness of the Helix. If $\alpha = 0$ the helix degenerates to a circle



Hence if the helix is too steep, the particle will slide because there is insufficient static friction to keep it stationary



$\tan(\gamma) = \alpha$.

Common Errors:

1. The first common error was to assume that R and α were not constants. The algebra to determine $\underline{v}$ and $\underline{a}$ then becomes so complicated in (c) that seeing the solution to the rest of the problem is very difficult.

2. A second common error is to assume that $\underline{F}_f = -\mu_s \|\underline{N}\|$ for static friction.

3. A third common error was to state that $\underline{e}_n = -\underline{e}_r$ by inspection.