Second Midterm Examination
Wednesday April 6 2011
Closed Books and Closed Notes

Question 1 *Planar Motion of a System of Two Particles* *(20 Points)*

As shown in Figure 1, a particle of mass m_1 is at rest and is attached to a fixed point O by a linear spring of stiffness K and unstretched length L_0. At time t $= 0$, a particle of mass m_2 traveling with a velocity vector $v_0\mathbf{E}_x$ impacts the particle of mass m_1. After the collision both particles adhere to each other, and can be considered as a particle of mass $m_1 + m_2$.

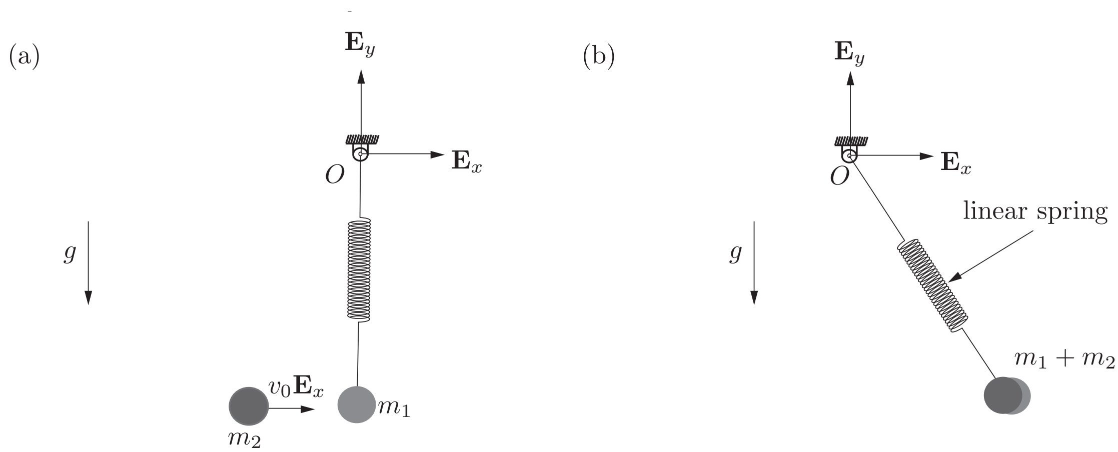


Figure 1: A system of two particles: (a) Prior to impact at $t = 0$, and (b) following the impact.

(a) *(4 Points)* Starting from the representation

$$\mathbf{r}_1 = r\mathbf{e}_r, \tag{1}$$

where $\mathbf{e}_r$ is a unit vector pointing from O along the spring to m_1, establish representations for the linear momentum $\mathbf{G}$, kinetic energy T, and acceleration $\mathbf{a}$ of the particle of mass $m_1 + m_2$ after the collision.

(b) *(4 Points)* Show that the velocities of the particle of mass $m_1 + m_2$ immediately following the collision are

$$\dot{r}(t = 0) = 0, \qquad r_0\dot{\theta}(t = 0) = \frac{m_2}{m_1 + m_2}v_0. \tag{2}$$

(c) *(4 Points)* Verify that the kinetic energy of the system is not conserved during the collision.

(d) *(4 Points)* Draw a freebody diagram of the particle of mass $m_1 + m_2$ following the collision. Give a clear expression for the spring force acting on the particle.

(e) *(4 Points)* Consider the system after impact. Starting from $\dot{T} = \mathbf{F} \cdot \mathbf{v}$ for a single particle, show that the total energy E of the particle of mass $m_1 + m_2$ is conserved. In your solution, give a clear expression for E.

Question 2 *A Double Pendulum* *(30 Points)*

As shown in Figure 2, a mechanical system consists of two particles. The particle of mass m_2 is connected using a pin joint and a rod of length L_1 to the particle of mass m_1. The particle of mass m_1 is attached by linear spring of stiffness K and unstretched length L_0 to a fixed point O. Both particles move on a smooth vertical plane.

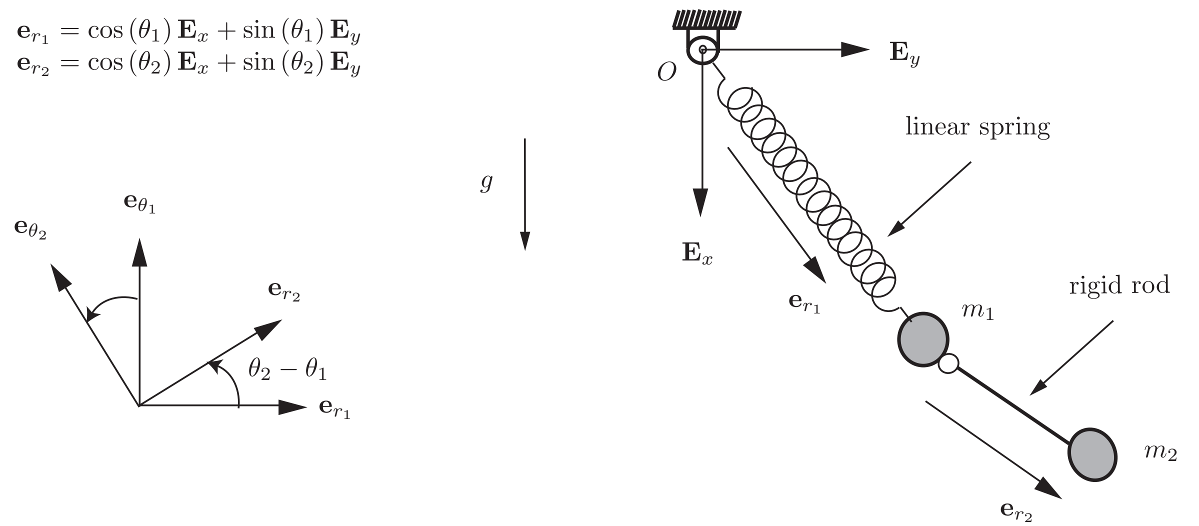


Figure 2: A system of two particles in motion on a smooth vertical plane.

(a) *(4 Points)* Starting from the representations for the position vectors of m_1 and m_2:

$$\mathbf{r}_1 = r\mathbf{e}_{r_1}, \qquad \mathbf{r}_2 = \mathbf{r}_1 + L_1\mathbf{e}_{r_2}, \tag{3}$$

establish an expression for the position vector $\mathbf{r}$ of the center of mass of the system. In addition, establish an expression for the linear momentum $\mathbf{G}$ of the system.

(b) *(8 Points)* Show that the kinetic energy of the system has the representation

$$T = \frac{m_1 + m_2}{2}\left(\dot{r}^2 + r^2\dot{\theta}_1^2\right) + \frac{m_2}{2}L_1^2\dot{\theta}_2^2 + \text{missing terms.} \tag{4}$$

For full credit supply the missing terms. (Hint: Notice the definition of $\mathbf{e}_{\theta_2}$ shown in Figure 2.)

(c) *(6 Points)* Draw 3 free-body diagrams: one for each of the individual particles and one for the system of particles. In your solution, give clear expressions for the spring force and tension force.

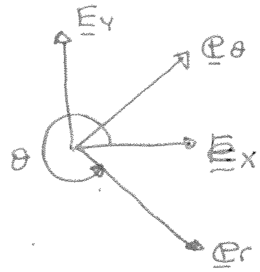
(d) *(7 Points)* Using the angular momentum theorem, show that $\dot{\mathbf{H}}_O \cdot \mathbf{E}_z$ depends entirely on the moments due to the gravitational forces on the particles. (Hint: use the identity $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{E}_z = (\mathbf{b} \times \mathbf{E}_z) \cdot \mathbf{a}$).

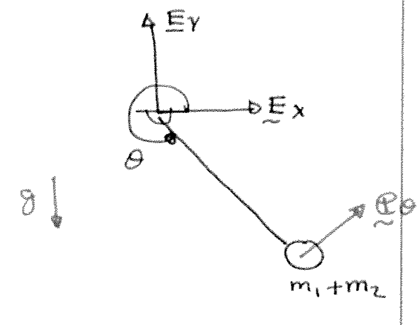
(e) *(5 Points)* Give an expression for the total energy E of the system of particles. Then, starting from the work-energy theorem for a system of particles,

$$\dot{E} = \mathbf{F}_{nc_1} \cdot \mathbf{v}_1 + \mathbf{F}_{nc_2} \cdot \mathbf{v}_2, \tag{5}$$

show that E is conserved.

QUESTION 2





(a) $\underline{r} = r\underline{e}_r \Rightarrow (m_1+m_2)\underline{v} = \underline{G} = (m_1+m_2)(\dot{r}\underline{e}_r + r\dot{\theta}\underline{e}_\theta)$

$T = \frac{1}{2}(m_1+m_2)(\dot{r}^2 + r^2\dot{\theta}^2)$

$\underline{a} = (\ddot{r} - r\dot{\theta}^2)\underline{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\underline{e}_\theta$

(b) During Collision $\underline{G}$ is conserved. at instant of collision $\underline{e}_r = -\underline{E}_Y$
$\underline{e}_\theta = \underline{E}_X$

Hence $(m_1+m_2)(-\dot{r}\underline{E}_Y + r\dot{\theta}\underline{E}_X) = m_2 v_0 \underline{E}_X$

$\Rightarrow \dot{r}(0) = 0 \qquad r_0\dot{\theta}(0) = \left(\frac{m_2}{m_1+m_2}\right) v_0$

(c) $T_1 = T$ before Collision $= \frac{1}{2} m_2 v_0^2$

$T_2 = T$ after collision $= \frac{1}{2}(m_2+m_1) r_0^2 \dot{\theta}^2(0) = \frac{1}{2}\frac{m_2^2}{m_1+m_2} v_0^2$

$T_1 - T_2 = \frac{1}{2} m_2 v_0^2 \left(1 - \frac{m_2}{m_1+m_2}\right) = \frac{1}{2}\frac{m_1 m_2}{m_1+m_2} v_0^2$

$\Rightarrow$ as $T_1 > T_2$, energy is lost during the collision.

(d)

$\underline{F}_s = -K(r-L)\underline{e}_r$

$N\underline{E}_z$ $\qquad -(m_1+m_2)g\underline{E}_Y$

$N\underline{E}_z$ is optional here (no credit lost or gained)

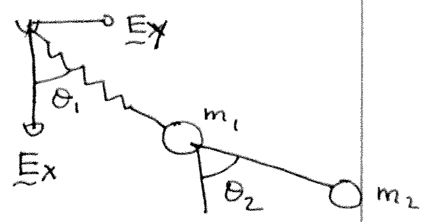
(e) $\dot{T} = \underline{F}\cdot\underline{v} = N\underline{E}_z\cdot\underline{v} - (m_1+m_2)g\underline{E}_Y\cdot\underline{v} + \underline{F}_s\cdot\underline{v}$

$= 0 - \frac{d}{dt}\left((m_1+m_2)g\underline{E}_Y\cdot\underline{r}\right) - \frac{d}{dt}\left(U_s = \frac{1}{2}K(r-L_0)^2\right)$

$\Rightarrow \frac{d}{dt}\left(E = T + (m_1+m_2)g\underline{E}_Y\cdot\underline{r} + U_s\right) = 0$

$\Rightarrow$ E is conserved.

QUESTION 2



a) $\underline{r}_1 = r\,\underline{e}_{r_1} \qquad \underline{r}_2 = \underline{r}_1 + L_1\,\underline{e}_{r_2}$

$$\underline{r} = \frac{m_1\underline{r}_1 + m_2\underline{r}_2}{m_1+m_2} = \underline{r}_1 + \left(\frac{m_2 L_1}{m_1+m_2}\right)\underline{e}_{r_2}$$

$$\underline{G} = m\underline{v} = m\dot{\underline{r}} = (m_1+m_2)\dot{\underline{r}}_1 + m_2 L_1\dot{\theta}_2\underline{e}_{\theta_2}$$

$$= (m_1+m_2)(\dot{r}\,\underline{e}_{r_1} + r\dot{\theta}_1\underline{e}_{\theta_1}) + m_2 L_1\dot{\theta}_2\underline{e}_{\theta_2}$$

b)
$$T = \tfrac{1}{2}m_1\underline{v}_1\cdot\underline{v}_1 + \tfrac{1}{2}m_2\underline{v}_2\cdot\underline{v}_2$$

$$= \tfrac{1}{2}m_1\underline{v}_1\cdot\underline{v}_1 + \tfrac{1}{2}m_2(\underline{v}_1 + L_1\dot{\theta}_2\underline{e}_{\theta_2})\cdot(\underline{v}_1 + L_1\dot{\theta}_2\underline{e}_{\theta_2})$$

$$= \tfrac{1}{2}(m_1+m_2)\underline{v}_1\cdot\underline{v}_1 + \tfrac{1}{2}m_2L_1^2\dot{\theta}_2^2 + \tfrac{1}{2}(2m_2)(L_1\dot{\theta}_2\underline{e}_{\theta_2}\cdot\underline{v}_2)$$

Now
$$\underline{v}_1 = \dot{r}\,\underline{e}_{r_1} + r\dot{\theta}_1\underline{e}_{\theta_1} \Rightarrow \underline{v}_1\cdot\underline{v}_1 = \dot{r}^2 + r^2\dot{\theta}_1^2$$

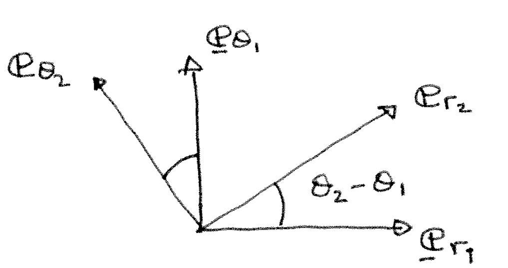
$$\underline{v}_1\cdot\underline{e}_{\theta_2} = \dot{r}\,\underline{e}_{r_1}\cdot\underline{e}_{\theta_2} + r\dot{\theta}_1\underline{e}_{\theta_2}\cdot\underline{e}_{\theta_1}$$

$$= -\dot{r}\sin(\theta_2-\theta_1) + r\dot{\theta}_1\cos(\theta_2-\theta_1)$$

Hence

$$T = \tfrac{1}{2}(m_1+m_2)(\dot{r}^2 + r^2\dot{\theta}_1^2) + \tfrac{1}{2}m_2L_1^2\dot{\theta}_2^2$$
$$+ m_2L_1\dot{\theta}_2\left(r\dot{\theta}_1\cos(\theta_2-\theta_1) - \dot{r}\sin(\theta_2-\theta_1)\right)$$

The ~~extra~~ last term comes from $2m_2\underline{v}_1\cdot L_1\dot{\theta}_2\underline{e}_{\theta_2}$

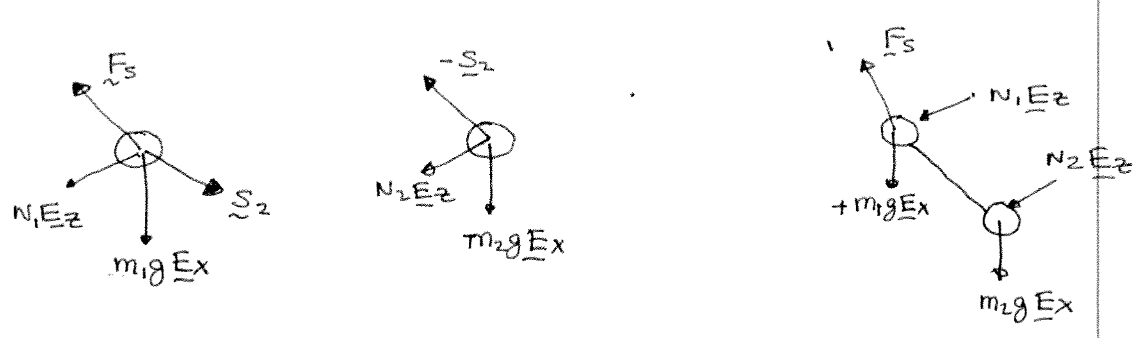


$$\underline{e}_{\theta_2} = \cos(\theta_2-\theta_1)\underline{e}_{\theta_1} - \sin(\theta_2-\theta_1)\underline{e}_{r_1}$$

The main error made in this problem was to assume $T = \tfrac{1}{2}(m_1+m_2)\underline{v}\cdot\underline{v}$

This assumption is false $\quad T \neq \tfrac{1}{2}(m_1+m_2)\underline{v}\cdot\underline{v}$

c)



$$\mathbf{F}_s = -K(r - L_0)\,\mathbf{e}_{r_1}$$

$$\mathbf{S}_2 = S_2\,\mathbf{e}_{r_2} \quad = \text{Tension force in Rod.}$$

d)

$$\dot{\mathbf{H}}_0 \cdot \mathbf{E}_z = (\mathbf{r}_1 \times \mathbf{F}_1)\cdot\mathbf{E}_z + (\mathbf{r}_2 \times \mathbf{F}_2)\cdot\mathbf{E}_z$$

$$= \left(\mathbf{r}_1 \times (\mathbf{F}_s + N_1\mathbf{E}_z + m_1 g\,\mathbf{E}_x)\right)\cdot\mathbf{E}_z$$

$$+ \mathbf{r}_2 \times (N_2\mathbf{E}_z + m_2 g\,\mathbf{E}_x)\cdot\mathbf{E}_z \quad + (\mathbf{r}_1 \times \mathbf{S}_2)\cdot\mathbf{E}_z$$

$$- (\mathbf{r}_2 \times \mathbf{S}_2)\cdot\mathbf{E}_z$$

$$= (\mathbf{r}_1 \times m_1 g\,\mathbf{E}_x)\cdot\mathbf{E}_z + (\mathbf{r}_2 \times m_2 g\,\mathbf{E}_x)\cdot\mathbf{E}_z$$

Notice how the moments due to $\mathbf{S}_2$ and $-\mathbf{S}_2$ cancel

$$+ \quad 0 \qquad (\mathbf{r}_1 \parallel \mathbf{F}_s \;;\; \mathbf{r}_1 - \mathbf{r}_2 \parallel \mathbf{S}_2 \;;\; \mathbf{r}_1 \times N_1\mathbf{E}_z \perp \mathbf{E}_z, \; (\mathbf{r}_2 \times N_2\mathbf{E}_z) \perp \mathbf{E}_z)$$

$$= -(m_1 g\,\mathbf{E}_y \cdot \mathbf{r}_1 + m_2 g\,\mathbf{E}_y \cdot \mathbf{r}_2)$$

$$= -m_1 g \sin\theta_1 r - m_2 g\,(r\sin\theta_1 + L\sin\theta_2)$$

$$\neq 0$$

$\Rightarrow$ $\mathbf{H}_0 \cdot \mathbf{E}_z$ is not conserved.

e)

$$E = T + \tfrac{1}{2}K(r - L_0)^2 - m_1 g\,\mathbf{E}_x \cdot \mathbf{r}_1 - m_2 g\,\mathbf{E}_x \cdot \mathbf{r}_2$$

$$\dot{E} = N_1\mathbf{E}_z \cdot \mathbf{v}_1 + N_2\mathbf{E}_z \cdot \mathbf{v}_2 + \mathbf{S}_2 \cdot (\mathbf{v}_1 - \mathbf{v}_2)$$

$$= 0 + 0 + 0 \qquad \left|\; \begin{array}{l} \mathbf{S}_2 \perp \mathbf{v}_1 - \mathbf{v}_2 \\ \mathbf{E}_z \perp \mathbf{v}_1 \\ \mathbf{E}_z \perp \mathbf{v}_2 \end{array}\right.$$

$\Rightarrow$ E is conserved.

Notice the expression for the gravitational potential energy $-m_1 g\,\mathbf{E}_x \cdot \mathbf{r}_1 - m_2 g\,\mathbf{E}_x \cdot \mathbf{r}_2$ $= -(m_1 + m_2) g\,\mathbf{E}_x \cdot \mathbf{r}$ where $\mathbf{r}$ is the position vector of the center of mass.