Math 104: Midterm 1 solutions

1. Consider the two sets

$$A = (0,1] \cup [4,\infty), \qquad B = \left\{ \frac{1}{2n} \; : \; n \in \mathbb{N} \right\}.$$

For each set, determine its maximum and minimum if they exist. For each set, determine its supremum and infimum. Detailed proofs are not required, but you should justify your answers.

Answer: Since the lower limit of A is an open interval, it does not have a minimum, however $\inf A = 0$. Since A is not bounded above, it does not have a maximum. $\sup A = \infty$ for sets not bounded above.

Since B has no smallest element, the minimum does not exist. However, since the fractions become arbitrarily close to 0, $\inf B = 0$. The maximum is given by $\max B = 1/2$, attained for the case when $n = 1$, and hence $\sup B = \max B = 1/2$.

2. Consider the following series, defined for $n \in \mathbb{N}$:

$$\sum \frac{6^n}{n^n}, \qquad \sum \frac{1}{n + 1/2}.$$

For each series, determine whether it converges or diverges. If you make use of any of the theorems for determining series properties, you should state which one you use.

Answer: For the first sequence, make use of the root test where $a_n = 6^n/n^n$. Then

$$(a_n)^{1/n} = \frac{6}{n}$$

which converges to zero as $n \to \infty$. Hence $\sum 6^n/n^n$ converges. For the second sequence, since $n + 1/2 \leq 2n$ for all $n \in \mathbb{N}$, then

$$\frac{1}{n + 1/2} \geq \frac{1}{2n}$$

for all $n \in \mathbb{N}$. Since $\sum \frac{1}{n}$ diverges, so does $\sum \frac{1}{2n}$, and hence by the comparison test, $\sum 1/(n + 1/2)$ does also.

3. Let S be a non-empty bounded subset of $\mathbb{R}$. Define $T = \{|x| : x \in S\}$ to be the set of all absolute values of elements in S. Prove that $\sup T = \max\{\sup S, -\inf S\}$.

Answer: Choose an element $t \in T$. Then either

- $t \in S$. Hence $t \leq \sup S$.
- There exists $s \in S$ such that $s = -t$. Hence $s \geq \inf S$, and therefore $t \leq -\inf S$.

Thus either $t \leq \sup S$ or $t \leq -\inf S$ so $t \leq \max\{\sup S, -\inf S\}$. Hence it is an upper bound.

Now suppose that l is an upper bound for T. Then $l \geq t$ for all elements $t \in T$. Hence $l \geq |s|$ for all elements $s \in S$, and thus

$$-l \leq s \leq l$$

for all elements in s, from which the following two deductions can be made:

- Since $s \leq l$ for all s, then $l \geq \sup S$ since $\sup S$ is the least upper bound for S.
- Since $-l \leq s$ for all s, then $-l \leq \inf S$ since $\inf S$ is the greatest lower bound for S. Hence $l \geq -\inf S$.

These two results show that $l \geq \max\{\sup S, -\inf S\}$. Hence $\max\{\sup S, -\inf S\}$ is an upper bound for T and it is the least upper bound, so it must be $\sup T$.

4. Let (s_n) and (t_n) be two sequences defined for $n \in \mathbb{N}$. Suppose $\lim s_n = \infty$, and $\limsup t_n < 0$. Prove that $\lim s_n t_n = -\infty$.

   Note: make sure to consider both cases when $\limsup t_n$ *is a real number, and when* $\limsup t_n$ *is* $-\infty$.

Answer: Define $v_N = \sup\{s_n : n > N\}$. There are two cases:

- $\limsup t_n = -q$ for some $q > 0$. Then there exists a K_1 such that $|v_N - (-q)| < q/2$ for all $N > K_1$. Hence $v_{K_1+1} < (-q) + (q/2) = -q/2$, and thus $t_n < -q/2$ for all $n > K_1 + 1$. For this case, define $\lambda = -q/2$.
- $\limsup t_n = -\infty$. Then there exists a K_1 such that $v_N < -1$ for all $N > K_1$. Hence $v_{K_1+1} < -1$, and thus $t_n < -1$ for all $n > K_1 + 1$. For this case, define $\lambda = -1$.

Now consider the sequence $s_n t_n$. Pick $M < 0$. Then since $\lim s_n = \infty$, there exists a K_2 such that $n > K_2$ implies that $s_n > M/\lambda$.

Now suppose $n > \max\{K_1 + 1, K_2\}$. Then $s_n > M/\lambda$ and $t_n < \lambda$, so $s_n t_n < M$. This is true for any $M < 0$, so $\lim s_n t_n = -\infty$.