Tuesday, February 21, 2005

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# ME 106
# Mid-term Exam I

Consider the hinged gate shown in the figure. On the right side, the fluid has constant density $\rho_R$ and has a free upper surface at $z = 0$, open to the atmospheric pressure $P_{\text{atm}}$. On the left side of the gate the fluid has constant density equal to that of water $\rho_w$ and has a free upper surface at $z = -H$, open to the atmospheric pressure $P_{\text{atm}}$. **It is important to note that the surfaces on the left and right sides of the gate are not the same.** A non-movable vertical segment extends from high above down to the hinge at point **A**. The gate, which extends from **A** to **B** is a semi-cylindrical plate (of negligible thickness and with no mass) such that its diameter is $D$ and both points **A** and **B** are at $z = -N$. There is a frictionless, fluid-tight seal between the right side of the tank and the point **B** on the hinged gate. The gate extends in the $y$-direction (into the paper) for a distance $L$ (where there are sidewalls in the $x$-$z$ planes with frictionless seals). The geometry of the gate (and every other property of the gate) is independent of $y$. Gravity $g$ points downward in the negative $z$ direction.

Our goal is to find the specific gravity of the fluid on the right such that there is no net torque on the gate.

**Remember, there is no partial credit for answers with incorrect dimensions.**

Find the pressures on the right and left side of the gate as functions of $z$. Is there any specific gravity the fluid on the right side could have such that the pressures on the right and left sides of the gate are everywhere the same? Explain.

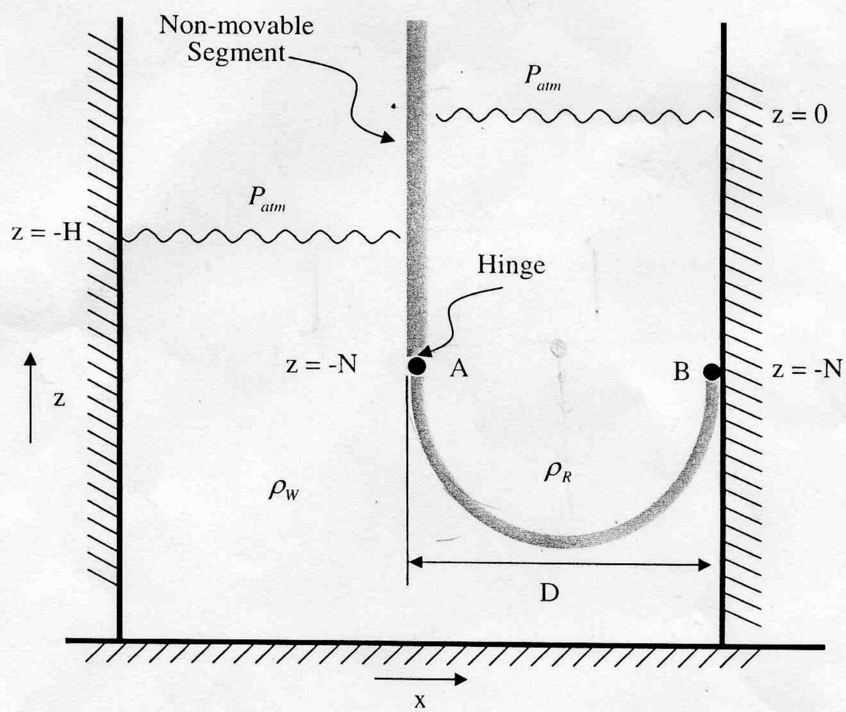
Find the force $\mathbf{F}_w$ on the bottom of the gate due to the pressure of the water. If you are clever you can compute the force using Archimedes' Principle applied to some volume and this will relieve you of doing any integrals.

Find the force $\mathbf{F}_R$ on the top of the gate due to the pressure of the fluid with density $\rho_R$. Again, if you are clever you can compute the force using Archimedes' Principle applied to some volume and this will relieve you of doing any integrals. **HINT**: The volume you use here may not be the same volume you used to find $\mathbf{F}_w$.

Find the torque due to the semi-cylindrical gate around the hinge at point **A**. Anticipating the fact that you want to minimize your work by first calculating the torque at a special point and then finding the torque around the hinge using the $\mathbf{r} \times \mathbf{F}$ rule, is there a location on the cylindrical gate where we can calculate our torque, that due to symmetry, makes the calculation of torque particularly easy?

Find an expression for the total torque around the hinge (it is a a vector!) in terms of $D$, $H$, $N$, $L$, $\rho_w$, $\rho_R$, $g$, and $P_{\text{atm}}$. Does your answer have the correct dimensions? Does the torque vector point in the direction and have the sign that common sense suggests? In the special case $H = 0$ and $\rho_R = \rho_W$, what should the value of the torque be? Does your expression give this value for this special case?

For $H \neq 0$, what value(s) of specific gravity $\rho_R/\rho_w$, if any, is the torque zero?



## 1. Pressures on the right and left side of the gate

The pressure in the water as a function of z is found as follows:

$$\frac{dP_w(z)}{dz} = -\rho_w g \Rightarrow P_w(z) - P_w(z=-H) = \int_{z=-H}^{z} -\rho_w g \; dz$$

and

$$P_w(z=-H) = P_{atm}$$

$$P_w(z) = P_{atm} - \rho_w g(z+H) \tag{1.1}$$

Similarly the pressure in the unknown fluid can be found as follows:

$$\frac{dP_R(z)}{dz} = -\rho_R g \Rightarrow P_R(z) - P_R(z=0) = \int_{z=0}^{z} -\rho_R g \; dz$$

and

$$P_R(z=0) = P_{atm}$$

$$P_R(z) = P_{atm} - \rho_R g z \tag{1.2}$$

It should be apparent that there is not a possible value for $\rho_R$ such that the pressures are the same on both sides of the gate everywhere. We have two linear equations for the pressure and in order to be the same they would require both the same slope and the same intercept. However, the air interfaces for the two fluids are at different heights so their intercepts differ even in the case that $\rho_w = \rho_R$ so the pressure never be everywhere the same on the right and the left sides of the gate.

## 2. The force $\mathbf{F}_w$ on the bottom of the gate

We can find the force acting on the gate by using Archimedes. Let us first find the force on the bottom side of the gate due to the pressure in the water. The easiest way of doing this is to consider a buoyancy force on half a cylinder and then subtract the pressure acting on the top of the geometry. Note that we can see by symmetry that there will be no force in the $\hat{\mathbf{x}}$ or $\hat{\mathbf{y}}$ directions.

$$\mathbf{B}_w = \rho_w g V \hat{\mathbf{z}} = \rho_w g L \frac{1}{2}\frac{\pi D^2}{4}\hat{\mathbf{z}} = \frac{\pi}{8}\rho_w g L D^2 \hat{\mathbf{z}}$$

The buoyancy force is the sum of the pressure force on the bottom of the half cylinder (the thing we want to find) and the pressure force on the top flat area, see figure 1. The top pressure force is simply the *water* pressure at that height times the area times the normal vector pointing out of the fluid:

$$\mathbf{F}_T = P_w(z=-N)(LD)(-\hat{\mathbf{z}})$$

$$P_w(z=-N) = \rho_w g(N-H) + P_{atm}$$

$$\mathbf{F}_T = -LDP_{atm}\hat{\mathbf{z}} - \rho_w g LD(N-H)\hat{\mathbf{z}}$$

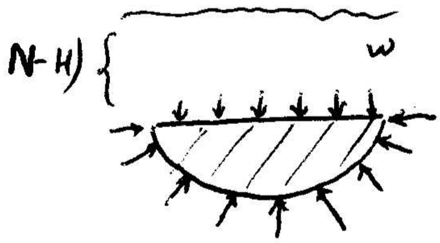



FIGURE 1. Buoyancy Force Due to Water

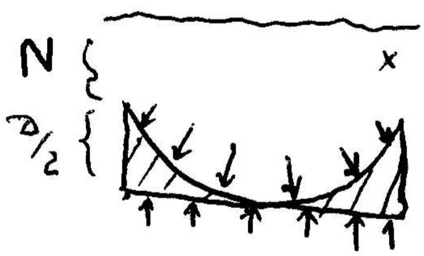



FIGURE 2. Buoyancy Force Due to Unknown Fluid

Note: water surface is at $z = -H$, so this height should be used to calculate the pressure on the top surface.

The buoyancy force is given by

$$\mathbf{B}_w = \mathbf{F}_w + \mathbf{F}_T,$$

so that the force on the bottom of the gate is given by

$$\mathbf{F}_w = \mathbf{B}_w - \mathbf{F}_T = LDP_{atm}\hat{\mathbf{z}} + \rho_w g LD\left(\frac{\pi D}{8} + N - H\right)\hat{\mathbf{z}} \tag{2.1}$$

## 3. THE FORCE $\mathbf{F}_R$ ON THE TOP OF THE GATE

We can use a similar technique to find the force due to the pressure of the unknown fluid on the top of the gate. In this case the volume used is not a half cylinder but rather a box with a half cylinder cut out of it. See figure 2.

$$\mathbf{B}_R = \rho_R g\left(LD\frac{D}{2} - L\frac{1}{2}\frac{\pi D^2}{4}\right)\hat{\mathbf{z}} = \rho_R g LD^2\left(\frac{1}{2} - \frac{\pi}{8}\right)\hat{\mathbf{z}}$$

This time the buoyancy force is given by

$$\mathbf{B}_w = \mathbf{F}_R + \mathbf{F}_B,$$

where $\mathbf{F}_B$ is given by the pressure force at the bottom of the volume in figure 2:

$$\mathbf{F}_B = P_R\left(z = -N - D/2\right)(LD)\left(-\hat{\mathbf{z}}\right),$$

where

$$P_R\left(z=-N-D/2\right)=\rho_w g\left(N+D/2\right)+P_{atm}$$

so that

$$\mathbf{F}_B=LDP_{atm}\hat{\mathbf{z}}+\rho_w gLD(N+D/2)\hat{\mathbf{z}}.$$

Note: water surface is at $z=0$, so this height should be used to calculate the pressure on the bottom surface.

The buoyancy force is given by

$$\mathbf{B}_R=\mathbf{F}_R+\hat{\mathbf{z}}P_R\left(z=-N-D/2\right)(LD),$$

so that the force on the top of the gate is

$$\mathbf{F}_R=\mathbf{B}_R-\mathbf{F}_B=\rho_R gLD\left(\frac{D}{2}-\frac{\pi D}{8}-N-\frac{D}{2}\right)\hat{\mathbf{z}}-LDP_{atm}\hat{\mathbf{z}}$$

This simplifies to

$$\mathbf{F}_R=-\rho_R gLD\left(N+\frac{\pi D}{8}\right)\hat{\mathbf{z}}-LDP_{atm}\hat{\mathbf{z}}. \tag{3.1}$$

Note: We could have found the same result for $\mathbf{F}_R$ by considering the weight of the unknown fluid pushing down on the gate plus the force due to the air pressure at the surface above the unknown fluid. The force acts in the $-\hat{\mathbf{z}}$ direction because the weight of the unknown fluid is pushing down on the gate.

What follows is the same solution by brute force integration over the cylindrical surface for those interested in comparing with that solution:

$$d\mathbf{F}_R=\hat{\mathbf{r}}P_R\left(z\right)\left(D/2\right)Ld\theta$$

where $\hat{\mathbf{r}}=\hat{\mathbf{x}}\cos\theta+\hat{\mathbf{z}}\sin\theta$ ,$z=-N+(D/2)\sin\theta$ and $\theta$ is from $\pi$ to $2\pi$

$$\mathbf{F}_R=\int_\pi^{2\pi}\hat{\mathbf{r}}P_R\left(z\right)\left(D/2\right)Ld\theta$$

$$\mathbf{F}_R=\int_\pi^{2\pi}\left(\hat{\mathbf{x}}\cos\theta+\hat{\mathbf{z}}\sin\theta\right)\left\{P_{atm}-\rho_R g\left[-N+(D/2)\sin\theta\ \right]\ \right\}\left(D/2\right)Ld\theta$$

$$\begin{aligned}\mathbf{F}_R\quad&=\quad\int_\pi^{2\pi}\left(\hat{\mathbf{x}}\cos\theta+\hat{\mathbf{z}}\sin\theta\right)\left(P_{atm}+\rho_R gN\right)\left(D/2\right)Ld\theta\\&\qquad-\int_\pi^{2\pi}\left(\hat{\mathbf{x}}\cos\theta\sin\theta+\hat{\mathbf{z}}\sin^2\theta\right)\rho_R g\left(D/2\right)^2Ld\theta\end{aligned}$$

and

$$\int_\pi^{2\pi}\cos\theta d\theta=0$$

$$\int_\pi^{2\pi}\sin\theta d\theta=-2$$

$$\int_\pi^{2\pi}\cos\theta\sin\theta d\theta=0$$

$$\int_\pi^{2\pi}\sin^2\theta d\theta=\pi/2$$

$$\begin{aligned}\mathbf{F}_R &= -\hat{\mathbf{z}}\{LD\left(p_{atm}+\rho_R gN\right)+L\rho_R g\left(D/2\right)^2\left(\pi/2\right)\} \\ &= -\hat{\mathbf{z}}LD\rho_R g\left(N+\pi D/8\right)-\hat{\mathbf{z}}LDP_{atm}\end{aligned}$$

## 4. Torque and specific gravity $\rho_R/\rho_w$

The center of force is, by symmetry, in the center of the gate: If you had a hinge at the center of the gate it would not experience any torques due to the fluids. We can therefore use the translation of torques rule from lecture 8.

First, we want to find the total force on the gate. This is simple:

$$\mathbf{F}=\mathbf{F}_R+\mathbf{F}_w=gLD\left[\rho_w\left(\frac{\pi D}{8}+N-H\right)-\rho_R\left(N+\frac{\pi D}{8}\right)\right]\hat{\mathbf{z}} \tag{4.1}$$

Note: the atmospheric pressure components canceled out as we would expect. We could as easily have used the gauge pressure instead of the absolute pressure in finding our forces. (This would not be true if the pressure of air was not taken to be constant over the distance H between the two interfaces.)

If $\rho_R=\rho_w$ and $H=0$, $\mathbf{F}=\mathbf{0}$. The torque will also be zero in this case. This is as we would expect because in this special case there is no change in the system if the gate is removed: there is water everywhere with the same pressure on both sides of the gate. ($\surd$)

We have only an $\hat{\mathbf{z}}$ component of the force. By symmetry in the $\hat{\mathbf{y}}$ direction we don't expect this force to produce any torques in the $\hat{\mathbf{x}}$ direction due to this force. Therefore, our torque around the gate will be only in the $\hat{\mathbf{y}}$ direction (which is the only direction allowed by the hinge in any case). We expect, by the right hand rule, that the water will be causing a torque in the $-\hat{\mathbf{y}}$ direction (out of the page) and the unknown fluid will be causing a torque in the $+\hat{\mathbf{y}}$ direction (into the page).

The vector from the hinge to the center of force is

$$\mathbf{r_{AC}}=\frac{D}{2}\hat{\mathbf{x}}-\frac{D}{2}\hat{\mathbf{z}}, \tag{4.2}$$

where $C$ is the center of force point.

$$\mathbf{\Gamma}=\mathbf{r}\times\mathbf{F}$$

$$\mathbf{\Gamma}=\left(\frac{D}{2}\hat{\mathbf{x}}-\frac{D}{2}\hat{\mathbf{z}}\right)\times gLD\left[\rho_w\left(\frac{\pi D}{8}+N-H\right)-\rho_R\left(N+\frac{\pi D}{8}\right)\right]\hat{\mathbf{z}}$$

$$\mathbf{\Gamma}=-\frac{1}{2}LD^2g\left[\rho_w\left(\frac{\pi D}{8}+N-H\right)-\rho_R\left(N+\frac{\pi D}{8}\right)\right]\hat{\mathbf{y}}$$

Finally, we have

$$\mathbf{\Gamma}=\frac{1}{2}LD^2g\left[\left(\rho_R-\rho_W\right)\left(\frac{\pi D}{8}+N\right)+\rho_W H\right]\hat{\mathbf{y}} \tag{4.3}$$

If $\rho_R=\rho_w$ and $H=0$, $\mathbf{\Gamma}=\mathbf{0}$. No torque when the unknown fluid is also water and the interface is everywhere at the same height. ($\surd$)

The torque is zero if

$$\mathbf{\Gamma}=\mathbf{0}\Rightarrow\frac{1}{2}LD^2g\left[\left(\rho_R-\rho_W\right)\left(\frac{\pi D}{8}+N\right)+\rho_W H\right]=0$$

$$\frac{\rho_R}{\rho_w} \equiv \gamma_R = 1 - \frac{H}{N + \frac{\pi D}{8}} = 1 - \frac{8H}{8N + \pi D} \tag{4.4}$$