Physics 137A: Midterm Solutions

February 23, 2011

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(2)

Let us first solve for the energy eigenstates with $E > 0$.

$$-\frac{\hbar^2}{2m}\psi'' = E\psi,\ x < -a$$

$$-\frac{\hbar^2}{2m}\psi'' = \left(E + \frac{40\hbar^2}{ma^2}\right)\psi,\ -a < x < 0$$

The general solution is

$$\psi(x) = \begin{cases} Ae^{ikx} + Be^{-ikx} & x < -a \\ Ce^{i\kappa x} + De^{-i\kappa x} & -a < x < 0 \\ 0 & x > 0 \end{cases}$$

in which $k = \sqrt{2mE}/\hbar$ and $\kappa = \sqrt{2m(E + 40\hbar^2/ma^2)}/\hbar$. Enforcing continuity of the wavefunction at $x = 0$ and $x = -a$ yields

$$C + D = 0$$

and

$$Ae^{-ika} + Be^{ika} = Ce^{-i\kappa a} + De^{i\kappa a} = C\left(e^{-i\kappa a} - e^{i\kappa a}\right)$$

respectively. We may also insist on continuity of the derivative at $x = -a$ (but not at $x = 0$). This gives

$$ik\left(Ae^{-ika} - Be^{ika}\right) = i\kappa\left(Ae^{-ika} + Be^{ika}\right)\frac{e^{-i\kappa a} - e^{i\kappa a}}{e^{-i\kappa a} + e^{i\kappa a}}$$

Since the scattering state wavefunction is non-normalizable, we cannot go any further. Note that all values of $E > 0$ are allowed.

Now let's consider solutions with $-\frac{40\hbar^2}{ma^2} < E < 0$. The general solution is

$$\psi(x) = \begin{cases} Ae^{kx} & x < -a \\ Ce^{i\kappa x} + De^{-i\kappa x} & -a < x < 0 \\ 0 & x > 0 \end{cases}$$

in which $k = \sqrt{-2mE}/\hbar$ and $\kappa = \sqrt{2m(40\hbar^2/ma^2 + E)}/\hbar$. Note that we have thrown out the solution which grows without bound in the $x < -a$ region. Once again, continuity of the wavefunction at $x = 0$ requires

$$C = -D$$

while continuity at $x = -a$ gives

$$Ae^{-ka} = C\left(e^{-i\kappa a} - e^{i\kappa a}\right)$$

so that the wavefunction takes the form

$$\psi(x) = \begin{cases} Ae^{kx} & x < -a \\ -A\frac{e^{-ka}}{\sin(\kappa a)}\sin(\kappa x) & -a < x < 0 \\ 0 & x > 0 \end{cases}$$

Normalization requires

$$\begin{aligned} \int_{-\infty}^{\infty} |\psi|^2 dx &= \int_{-\infty}^{-a} |A|^2 e^{2kx} dx + \int_{-a}^{0} |A|^2 \frac{e^{-2ka}}{\sin^2(\kappa a)} \sin^2(\kappa x) dx \\ &= e^{-2ka}\left(\frac{1}{2k} - \frac{\cot(\kappa a)}{2\kappa} + \frac{a}{2\sin^2(\kappa a)}\right) \\ \implies A &= \frac{\sqrt{2k\kappa}e^{ka}}{\sqrt{\kappa - k\cot(\kappa a) + ak\kappa\csc^2(\kappa a)}} \end{aligned}$$

Finally we must insist on continuity of the derivative at $x = -a$, which will fix the allowed values of the energy.

$$\begin{aligned} ke^{-ka} &= -\frac{\kappa e^{-ka}}{\sin(\kappa a)}\cos(-\kappa a) \\ \implies k &= -\kappa\cot(\kappa a) \end{aligned}$$

(3)

Plugging in the definitions of k and κ, we find

$$\tan(\sqrt{80}\sqrt{1-y}) = -\sqrt{\frac{1-y}{y}}$$

in which $y \equiv -\frac{ma^2E}{40\hbar^2}$. Note that $0 < y < 1$. Comparing both sides of the equation, we see that there are exactly 3 bound states (plot both sides of the equation to check).

(4)

The reflectivity for a scattering state coming in from $-\infty$ is nothing but $\frac{|B|^2}{|A|^2}$ from part (2), above. Recall that we found

$$\begin{aligned} ik\left(Ae^{-ika} - Be^{ika}\right) &= i\kappa\left(Ae^{-ika} + Be^{ika}\right)\frac{e^{-i\kappa a} - e^{i\kappa a}}{e^{-i\kappa a} + e^{i\kappa a}} \\ &= \kappa\left(Ae^{-ika} + Be^{ika}\right)\tan(\kappa a) \\ \implies \frac{ik}{\kappa}\left(1 - \frac{B}{A}e^{2ika}\right) &= \tan(\kappa a)\left(1 + \frac{B}{A}e^{2ika}\right) \\ \implies \frac{B}{A} &= e^{-2ika}\frac{ik - \kappa\tan(\kappa a)}{ik + \kappa\tan(\kappa a)} \end{aligned}$$

Thus we have

$$R = \frac{|B|^2}{|A|^2} = \frac{\kappa^2 \tan^2(\kappa a) + k^2}{\kappa^2 \tan^2(\kappa a) + k^2} = 1$$

which makes sense since there is an infinite barrier at $x = 0$. The probability to escape to $x = +\infty$ is zero, and the wavefunction has too much energy to be bound, so it must reflect back to $x = -\infty$ with 100% probability.

2

(1)

$$\begin{aligned} \frac{\partial}{\partial x} e^{ikx} (\tanh x + C) &= ike^{ikx} (\tanh x + C) + e^{ikx} \text{sech}^2 x \\ \implies \frac{\partial^2}{\partial x^2} e^{ikx} (\tanh x + C) &= -k^2 e^{ikx} (\tanh x + C) + ike^{ikx} \text{sech}^2 x - 2e^{ikx} \text{sech}^2 x \tanh x \end{aligned}$$

so that

$$\begin{aligned} -\frac{\partial^2 \psi}{\partial x^2} - 2\text{sech}^2 x \psi &= e^{ikx} \left[k^2 \tanh x + Ck^2 - ik\text{sech}^2 x + 2\text{sech}^2 x \tanh x - 2\text{sech}^2 \tanh x - 2C\text{sech}^2 x\right] \\ &= e^{ikx} \left[k^2 \tanh x + Ck^2 - ik\text{sech}^2 x - 2C\text{sech}^2 x\right] \\ &= k^2 \psi \end{aligned}$$

for $C = -ik/2$. Note that as $x \to -\infty$, this wavefunction becomes $\psi \to -(1 + ik/2)e^{ikx}$ a plane wave incident from the left. As $x \to \infty$, $\psi \to (1 - ik/2)e^{ikx}$, a plane wave travelling to the right. Thus the transmission coefficient is just

$$T = \frac{|1 - ik/2|^2}{|1 + ik/2|^2} = \frac{1 + k^2/4}{1 + k^2/4} = 1$$

$R + T = 1 \implies R = 0$.

(2)

Note that the above is a solution to the Schrodinger equation with energy $E = k^2$ for both k and $-k$. Thus the general solution is

$$\psi = Ae^{ikx} (\tanh x - ik/2) + Be^{-ikx} (\tanh x + ik/2)$$

We can get the bound state solutions by taking $k \to ik$ (equivalent to sending $E \to -E$). This gives the general bound-state solution

$$\psi = Ae^{-kx} (\tanh x + k/2) + Be^{kx} (\tanh x - k/2)$$

with energy $E = -k^2$. In the region $x < 0$ we may throw out the solution which grows unbounded as $x \to -\infty$, and of course we may perform the analogous cut on the $x > 0$ solution. This gives

$$\psi = \begin{cases} Ae^{kx} (\tanh x - k/2) & x < 0 \\ Be^{-kx} (\tanh x + k/2) & x > 0 \end{cases}$$

Continuity at $x = 0$ requires

$$A = -B$$

Continuity of the derivative requires

$$A(1 - k^2/2) = -A(1 - k^2/2)$$

Thus the only bound-state solution is $k^2 = 2 \implies E = -2$.

(3)

Since this is the unique bound-state wavefunction, it must also be the ground state. This is further confirmed by noting that the wavefunction has no nodes, and therefore must be the state of lowest energy.