2	3	4	5	6	Σ
12	3	11	14	10	62

Bianca 9-10

Math 53, F.Rezakhanlou
First Midterm, September 28, 2006

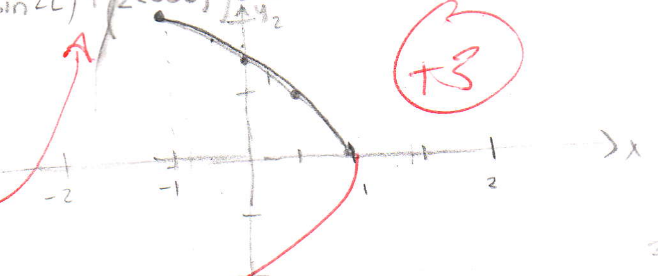
Each question should be answered directly. Use the back of these sheets if necessary. Justify your assertions; include detailed explanation, and show your work. Closed book exam, no sheet of notes and no calculator.

Your Name:
Your Section:

1. *(16 points)* (a) Graph the planar curve $x(t) = \cos 2t$, $y(t) = 2\sin t$ for $t \in \mathbb{R}$ and find the surface area generated by rotating the curve about the x-axis (do not evaluate the integral).

$SA = 2\pi y \int \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\, dt$ +1

$SA = 2\pi(2\sin t)\int_0^{\pi/2} \sqrt{(-2\sin 2t)^2 + (2\cos t)^2}\, dt$

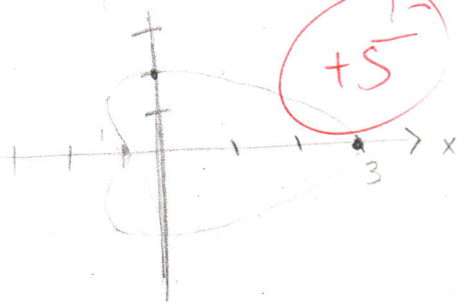


+3

$x = \cos 2t$ $\quad y = 2\sin t$

t	x	y
0	1	0
$\pi/6$	$\frac{1}{2}$	1
$\pi/4$	0	$\sqrt{2}$
$\pi/3$	$-1/2$	$\sqrt{3}$
$\pi/2$	-1	2
$2\pi/3$	$-1/2$	$\sqrt{3}$
$3\pi/4$	0	$\sqrt{2}$

(b) Graph the curve $r = 2 + \cos\theta$ and find its length (do not evaluate the integral).



+5

14/16

θ	r
0	3
$\pi/3$	2.5
$\pi/2$	2
$2\pi/3$	1.5
$3\pi/4$	$4-\frac{\sqrt{2}}{2}$
π	1
$5\pi/3$	1.5
$3\pi/2$	2

$L = \int_0^{2\pi} \sqrt{(2+\cos\theta)^2 + (-\sin\theta)^2}\, d\theta$ +3

$\cos 2\theta$

$2\cos\theta$

2|3|4|5|6| Σ

Bianca

2. *(14 points)* **(a) Find the range and domain of the function** $f(x, y, z) = \ln(4x^2 - y^2 - z^2)$.

12/14

7/7

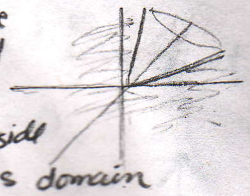
$D = \{(x,y,z) \mid 4x^2 - y^2 - z^2 > 0\} = \{(x,y,z) \mid x^2 > \frac{y^2+z^2}{4}\}$

$R =$ all real #

$(-\infty, \infty)$

$4x^2 > y^2 + z^2$ = cone centered around pos x axis

all area outside cone is domain

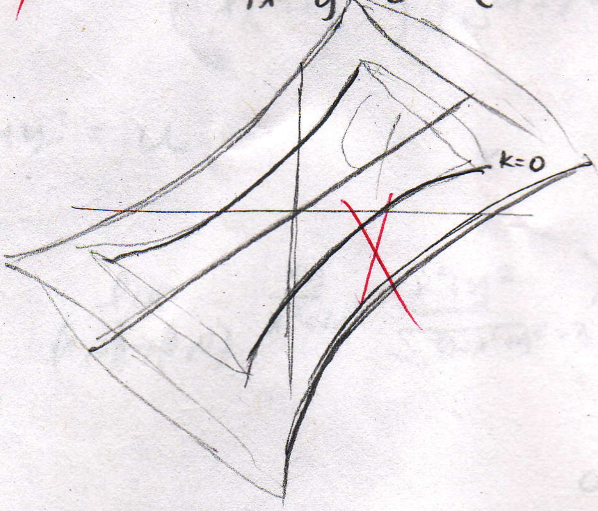


(b) Find the level surfaces of f.

5/7

$\ln(4x^2 - y^2 - z^2) = k$

$4x^2 - y^2 - z^2 = e^k$



$k = 0$

$4x^2 - y^2 - z^2 = 1$ = hyperboloid of 2 sheets

$y^2 + z^2 = -1$

centered on x-axis

$k = 1$

$4x^2 - y^2 - z^2 = e$

$\frac{4x^2}{e} - \frac{y^2}{e} - \frac{z^2}{e} = 1$

$\sqrt{e}$

$\sqrt{e} > 1$

3

3. *(6 points)* Find

$$\lim_{(x,y)\to(0,0)} \left(\frac{xy}{x^2+2y^2} + \frac{x^2+y^2}{\sqrt{9+x^2+y^2}-3} \right), = \lim_{(x,y)\to(0,0)} \left(\frac{xy}{x^2+2y^2}\right) + \lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{9+x^2+y^2}-3}$$

if the limit exists, or show that the limit does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+2y^2} \Rightarrow \overset{x=y}{\lim_{x\to 0}} \frac{x^2}{x^2+2x^2} = \lim_{x\to 0} \frac{x^2}{3x^2} = \frac{1}{3}$$

$$\overset{x=0}{\lim_{y\to 0}} \frac{0(y)}{0+(2y)} = \lim_{y\to 0} 0 = 0$$

limit doesn't exist

$$\lim_{r\to 0} \frac{r^2}{\sqrt{9+r^2}-3} = \frac{0}{\sqrt{9+0}-3} = \frac{0}{0} \Rightarrow \infty$$

$$\frac{2r}{\frac{1}{2}(}(9+r^2)^{1/2}$$

$$\frac{xy\left(\sqrt{9+x^2+y^2}-3\right) + (x^2+y^2)(x^2+2y^2)}{(x^2+2y^2)\left(\sqrt{9+x^2+y^2}-3\right)}$$

$$x^2+y^2 = u$$

$$x$$

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+2y^2} + \frac{x^2+y^2}{\sqrt{9+x^2+y^2}-3} > \lim_{x,y\to(0,0)} \frac{xy}{x^2+2y^2}$$

and $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+2y^2}$ does not exist

False

so limit of function doesn't exist.

4. *(15 points)* (a) Find the unit tangent vector $\mathbf{T}(t)$ for the curve $\mathbf{r}(t) = < \sin t, \cos t, e^t >$.

$$r'(t) = \langle \cos t, -\sin t, e^t \rangle$$

$$T(t) = \frac{r'(t)}{\|r'(t)\|} = \frac{\cos t\,\vec{i} - \sin t\,\vec{j} + e^t\,\vec{k}}{\sqrt{\cancel{\cos^2 t + \sin^2 t} + e^{2t}}}$$

$$= \frac{\cos t\,\vec{i} - \sin t\,\vec{j} + e^t\,\vec{k}}{e^t}$$

-1

$$= \frac{\cos t}{e^t}\vec{i} - \frac{\sin t}{e^t}\vec{j} + \vec{k}$$

(b) Find the parametric equation of the tangent at $\mathbf{r}(0)$.

$$r(0) = \langle \sin(0), \cos(0), e^0 \rangle$$
$$= \langle 0, 1, 1 \rangle$$
$$r' = \langle \cos t, -\sin t, e^t \rangle$$
$$r'(0) = \langle \cos 0, -\sin 0, e^0 \rangle = \langle 1, 0, 1 \rangle$$

$x = 0 + t \quad y = 1 \quad z = 1 + t$

$x = t \quad y = 1 \quad z = 1 + t$

(c) Show that $\mathbf{T}'(t) \cdot \mathbf{T}(t) = 0$. (Hint: Use $|\mathbf{T}(t)|^2 = 1$.)

$$T'(t) \cdot T(t) =$$
$$T(t) = \frac{\cos t}{e^t}\vec{i} - \frac{\sin t}{e^t}\vec{j} + \vec{k}$$
$$T'(t) = \frac{(-\sin t\,e^t) - (e^t \cos t)}{(e^t)^2}\vec{i} + \frac{(\cos t\,e^t + \sin t\,e^t)}{(e^t)^2} - 0$$

$$\cancel{(T'(t) \cdot T(t) = 0)^2} \qquad \cancel{(T'(t) \cdot T(t))^2} = 0$$

$$\cancel{T(t)^2 = 1} \qquad T'(t) \cdot T(t$$

-3

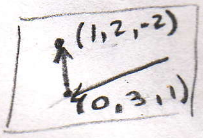
$$\frac{-\sin t - \cos t}{e^t}\vec{i} - \frac{\cos t + \sin t}{e^t}\vec{j}$$

$$T'(t) \cdot T(t) = \left(\frac{-\sin t - \cos t}{e^t}\right)\left(\frac{\cos t}{e^t}\right) + \left(\frac{-\cos t + \sin t}{e^t}\right)\left(\frac{-\sin t}{e^t}\right)$$

$$= \cancel{\frac{-\sin t \cos t}{(e^t)^2}} - \frac{\cos^2 t}{(e^t)^2} \cancel{+ \frac{\cos t \sin t}{(e^t)^2}} - \frac{\sin^2 t}{(e^t)^2} = -\frac{\cos^2 t + \sin^2 t}{(e^{2t})} = \frac{-1}{e^{2t}}$$

$$-\left(\frac{\cos^2 t}{(e^t)^2} + \frac{\sin^2 t}{(e^t)^2}\right) = -\left(\frac{1}{(e^t)^2} + \frac{1}{e^t}\right) = \frac{1}{(e^t)^2} + \frac{e^t}{(e^t)^2}$$

5. *(14 points)* (a) Find an equation for the plane through the point $(1, 2, -2)$ that contains the line $x = 2t$, $y = 3 - t$, $z(t) = 1 + 3t$.



$\vec{N} \perp$ to both lines so. $\vec{N} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 1 & 3 \\ 2 & -1 & 3 \end{vmatrix} = (3-(-3))\vec{i} - (-3-6)\vec{j} + (1-2)\vec{k}$

$= 6\vec{i} + 9\vec{j} - \vec{k}$

$(0,3,1) - (1,2,-2)$

$= \langle -1, 1, 3 \rangle$ $\quad \langle -1,1,3\rangle \times \langle 2,-1,3\rangle$

$0-1=-1$

$3-2=1$

$1+2=3$

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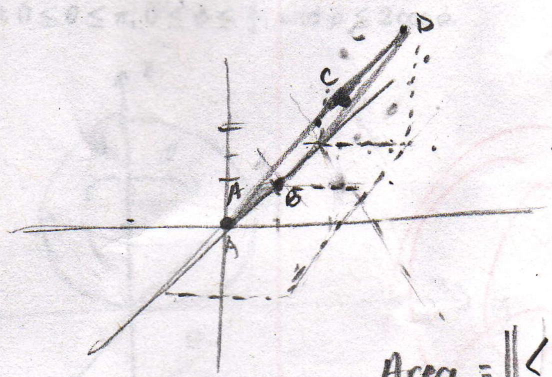
Plane $= 6(x-1) + 9(y-2) - 1(z+2) = 0$

$6x - 6 + 9y - 18 - z - 2 = 0$

$6x + 9y - z = 18 + 2 + 6$

$6x + 9y - z = 26$

(b) Find the area of the parallelogram with vertices $A = (0, 0, 0)$, $B = (1, 0, 0)$, $C = (1, 2, 2)$, $D = (2, 2, 2)$.



$AB = (1,0,0)$

$AC = (1,2,2)$

Area $= \| \langle 1,0,0\rangle \times \langle 1,2,2\rangle \|$

$= \left\| \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 1 & 2 & 2 \end{vmatrix} \right\|$

$= (0-0)\vec{i} - (2-0)\vec{j} + (2-0)\vec{k}$

$= 0 - 2\vec{j} + 2\vec{k}$

$= \sqrt{2^2 + 2^2} = \sqrt{4+4}$

$= \sqrt{8}$

$= 2\sqrt{2}$

7

6. (a) *(4 points)* Write the equation $x^2+y^2+z^2=3z$ is cylindrical and spherical coordinates.

$x^2+y^2+z^2=\rho^2 \qquad z=\rho\cos\phi$

so $x^2+y^2+z^2=3z \Rightarrow \rho^2=3\rho\cos\phi$

$\rho=3\cos\phi$

$x^2+y^2=r^2$

so $x^2+y^2+z^2=3z = r^2+z^2=3z$

$r^2+(z^2-3z)=0$

$r^2+z(z-3)=0$

$-z(z-3)=r^2$

$z=r^2$ or $z=r^2+3$

4

(b) *(6 points)* Sketch or describe the solid consisting of all points with spherical coordinates (ρ,θ,ϕ) such that $0\le\theta\le\pi$, $0\le\phi\le\frac{\pi}{4}$, and $\rho\le 2\cos\phi$.

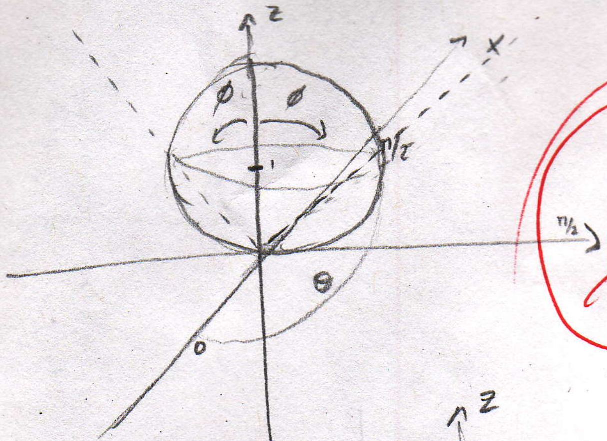
$\rho^2\le 2\rho\cos\phi$

$x^2+y^2+z^2\le 2z$

$x^2+y^2+z^2-2z\le 0$

$x^2+y^2+(z-1)^2\le 1$

sphere centered at $(0,0,1)$ w/ radius 1



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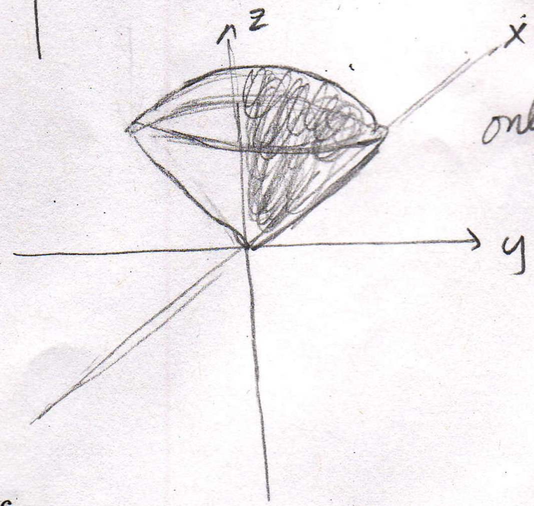

only the side on pos. y axis side due to ϕ and cone w/ ice cream