Friday, December 18, 12:30-3:30 PM, 2009.

Answer all questions for a maximum of 100 points. Please write all answers in the space provided. If you need additional space, use the reverse sides. Indicate your answer as clearly as possible for each question. Write your name at the top of each page as indicated. *Read each question carefully!*

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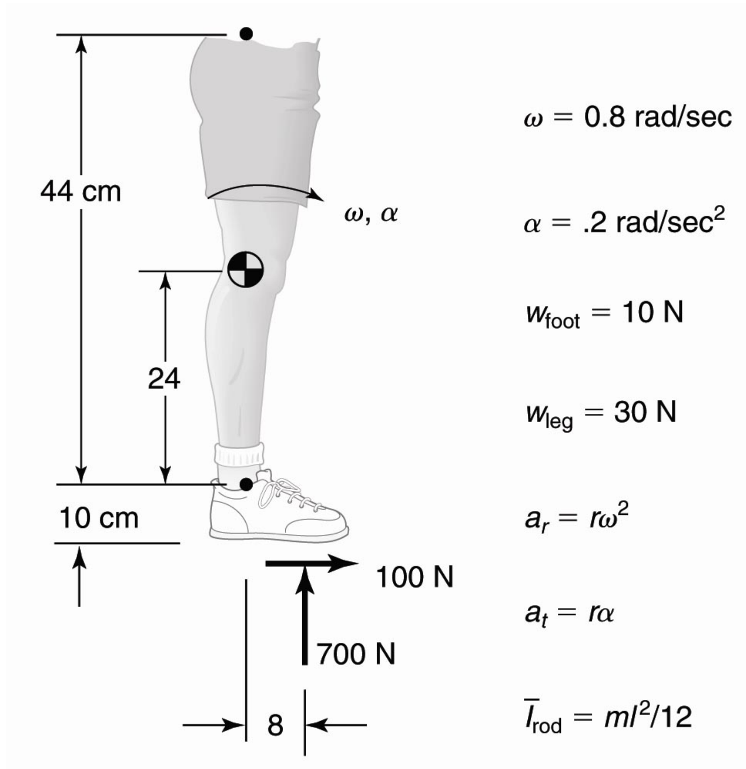
1. (30 points total) Dynamic Analysis

Here's a schematic of a leg at the instant when the foot is making contact with the ground during the mid-stance phase of gait. Assume at this instant that the foot is stationary and the leg (femur and tibia acting as a single rigid body) is rotating with angular velocity ω and acceleration α about a fixed point at the center of the ankle joint. The center of mass of the femur-tibia is directly over this point, as is the center of the hip joint, and the center of mass of the foot is directly below this point. The ground reaction force acts at the front of the foot, as shown.



(i) **[10 points]** Draw a free body diagram for the combined femur-tibia rigid body. Draw another free body diagram for the foot. In both free body diagrams, include all accelerations.

<u>FEMUR-TIBIA FBD:</u> <u>FOOT FBD:</u>

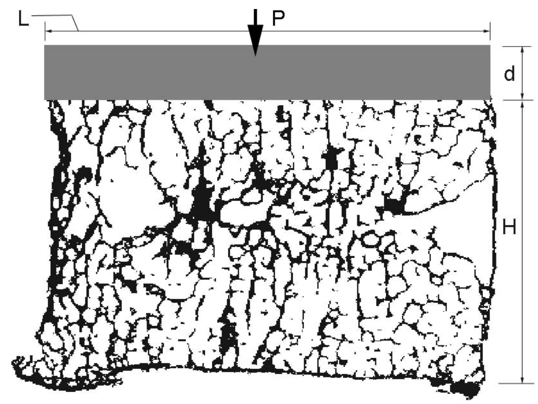
(ii) [5 points] For the data shown, and using a <u>static</u> analysis, calculate the magnitude and direction of the resultant moment at the hip joint.

(iii) [10 points] Instead, using a full dynamic analysis, calculate the magnitude and direction of the resultant moment at the hip joint. Assume the femur-tibia can be modeled as a slender rod of mass m and length l, having a moment of inertia about its mass center as shown in the schematic.

(iv) **[5 points]** Calculate the percent error in the magnitude of the resultant moment at the hip joint that is introduced when you use a static analysis instead of the more rigorous dynamic analysis. Based on your results, comment on the need for a fully dynamic analysis in gait studies.

2. (25 points total) Beam-on-Elastic-Foundation (BOEF) Theory

A sagittal slice from an elderly human vertebral body is shown at right, with a plastic (elastic modulus E_{UHMWPE}) component of a "total disc replacement" attached on top. The endplate has been removed during surgery and you can assume that the plastic implant is supported solely by the underlying trabecular bone (*i.e.* ignore any support by the very thin cortex). Vertical loading (point force P) from the articulating metal component (not shown) occurs at the mid point. Assume we can model this structure as a beam on an elastic foundation (BOEF), in which the plastic implant is the beam of thickness d and length L and the trabecular bone is a uniform elastic foundation of effective height H.



(i) **[7 points]** Derive the relationship between k (the "foundation modulus") and E_{TB} (the Young's modulus of the trabecular bone) in terms of the relevant dimensions. Assume that both the bone and the implant have a uniform depth b into the page.

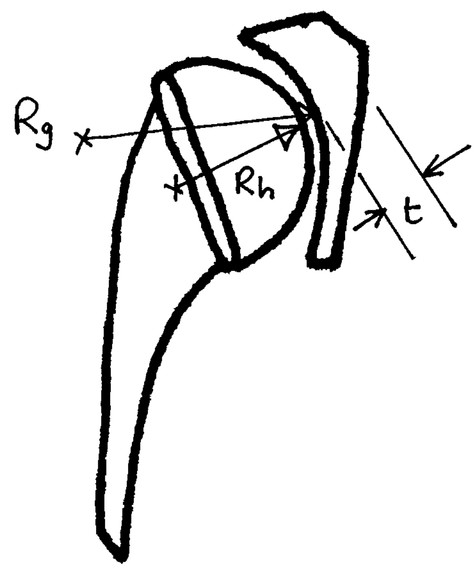
(ii) **[8 points]** In the context of BOEF theory, explain the concept of "rigid" versus "flexible" behaviors. Recall: $\lambda = \sqrt[4]{\dfrac{k}{4EI}}$

(iii) **[5 points]** If H = 30 mm, b = 30 mm, L = 40 mm, E_{UHMWPE}=1000 MPa, and E_{TB}=300 MPa, calculate the minimum value of plastic thickness required to ensure "rigid" behavior of the system.

(iv) **[5 points]** For such "rigid" behavior, and assuming that the maxiumum allowable compressive strain on the underlying trabecular bone is 1.0%, what is the maximum allowable force P that can safely be transmitted through this "total disc replacement" implant?

3. (20 points total) Contact Stresses and Design of Shoulder Prostheses

The schematic at right shows a total joint prosthesis for the gleno-humeral joint with the main design parameters: radius of the humeral R_h and glenoid R_g components and average thickness of the plastic, t.



(i) **[10 points]** Indicate whether the following statements are true or false:

The maximum compressive contact stresses will increase if:

a) R_h is increased (all else constant)

b) R_g is increased (all else constant)

c) t is increased (all else constant)

d) the elastic modulus of the glenoid (plastic) component is increased (all else constant)

e) the elastic modulus of the humeral component is increased (all else constant)

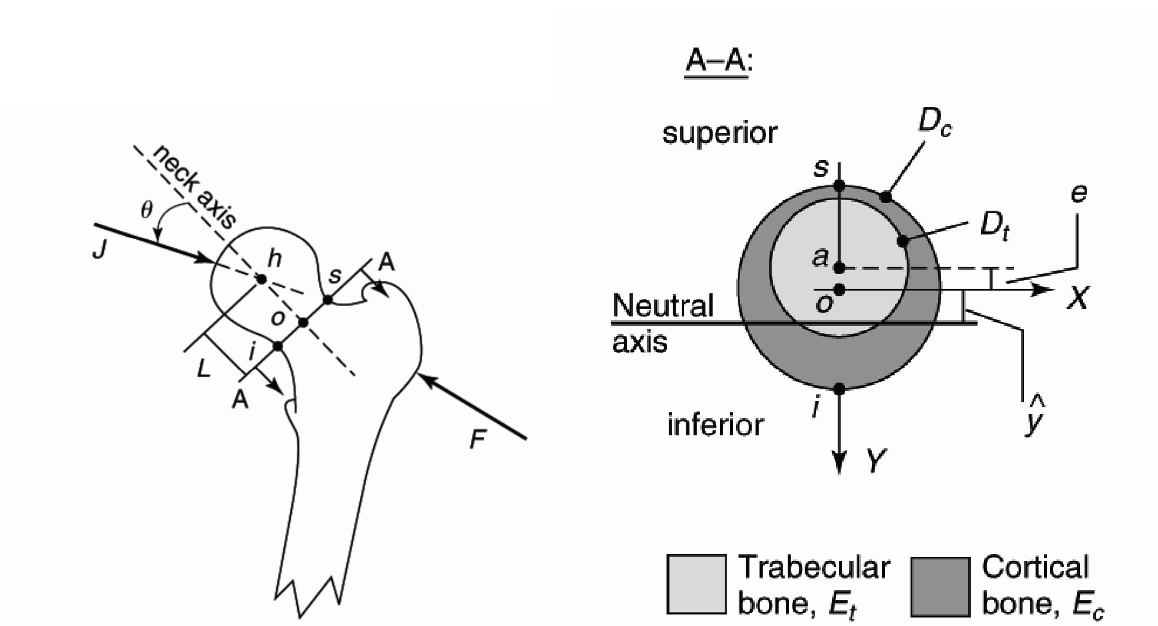
(ii) **[5 points]** Will the contact stresses be more sensitive to small (say ± 1 mm) changes in R_h or R_g? *Explain your answer*.

(iii) **[5 points]** State two pros and two cons of using metal-backing of plastic components in total joint replacement.

4. (25 points total) Composite Beam Theory and Hip Fracture Risk
Hip fracture is the most devastating type of osteoporotic fracture. Consider the situation, below, in which the impact force at the side of the hip during a fall is F, and that it develops a joint contact force J, at some angle θ as well as some equilibrating loads acting on the distal femur (not shown).

Section A-A is a schematic of the narrowest cross-section of the femoral neck. The cross-section contains circular areas representing the cortical bone (modulus E_c, center o, outer diameter D_c) and the trabecular bone (modulus E_t, center a, outer diameter D_t) with the trabecular region displaced superiorly by a distance e. The X-Y coordinate system is centered at origin o, and the neutral axis of the composite beam is at a distance $\hat{y}$ from the X-axis.

The angle of the joint contact force, θ, plays an important role in hip fracture etiology. The amount of eccentricity, e, between the cortical and trabecular bone also plays an important role. Let's investigate how the stress at the inferior-most point, i, on the femoral neck depends on θ and e.



The stress acting on the inferior-most point, i, can be expressed by the following composite beam theory-based equation:

$$\sigma = -\frac{PE_c}{\sum_i E_i A_i} + \frac{ME_c t}{\sum_i E_i \hat{I}_i}$$

in which P is the axial force acting on section A-A, M is the total bending moment acting on section A-A (this is the moment acting about the neutral axis), t is the distance from point i to the neutral axis of section A-A, and the summation terms ΣEA and $\Sigma E\hat{I}$ represent the axial and flexural stiffness, respectively, of the composite section, the latter with respect to the neutral axis.

Using the above diagram and stress equation, write out expressions for the following terms as a function only of the variables indicated:

Recall the following equations from beam theory $\hat{y} = \dfrac{\sum_i E_i A_i \bar{y}_i}{\sum_i E_i A_i}$ and $\bar{I}_{circle} = \dfrac{\pi D^4}{64}$

(i) **[5 points]** $P = f(J,\theta) =$

(ii) **[5 points]** $M = f(J,\theta,L,\hat{y}) =$

(iii) **[5 points]** $t = f(D_c, D_t, e, E_c, E_t) =$

(iv) **[5 points]** $\Sigma E\hat{I} = f(D_c, D_t, e, \hat{y}, E_c, E_t) =$

(v) **[5 points]** Using the stress equation from the problem statement and the relationships you derived in parts (i)-(iv), comment on the effect of <u>increasing</u> trabecular eccentricity on the stress at the inferior-most point of the femoral neck. Comment also on the effect of <u>increasing</u> the angle of the joint contact force.