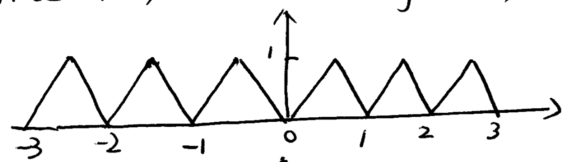
Midterm solution EE120

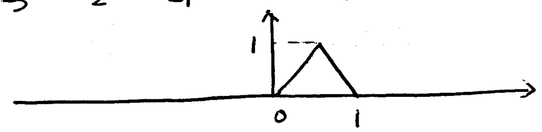
1. (a) let's denote the function in the problem to be $x(t)$

then we can write $x(t)$ as the sum of $x_1(t)$, $x_2(t)$

where $x_1(t) =$

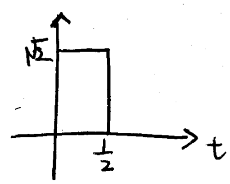


$x_2(t) =$



Now

$x_2(t) =$



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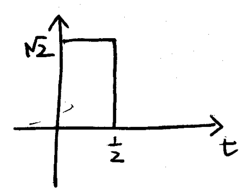


$$\Rightarrow \quad X_2(w) = \left(\frac{2\sqrt{2}\sin\frac{w}{4}}{w}e^{-j\frac{w}{4}}\right)^2 = 8\left(\frac{\sin\frac{w}{4}}{w}\right)^2 e^{-j\frac{w}{2}}$$

Since $x_1(t) = x_2(t) * \sum_{k=-\infty}^{\infty}\delta(t-k)$

$$X_1(w) = 8\left(\frac{\sin\frac{w}{4}}{w}\right)^2 e^{-j\frac{w}{2}} \cdot 2\pi\sum_{k=-\infty}^{\infty}\delta(w-2\pi k)$$

$$= \sum_{k=-\infty}^{\infty} 8\left(\frac{\sin\frac{2\pi k}{4}}{2\pi k}\right)^2 e^{-j\pi k} \cdot 2\pi\delta(w-2\pi k)$$

$$= \sum_{k=-\infty}^{\infty}\frac{4}{\pi}\left(\frac{\sin\frac{\pi k}{2}}{k}\right)^2(-1)^k\delta(w-2\pi k)$$

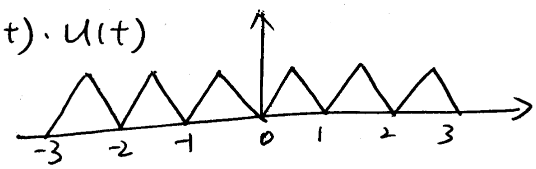
$$= \pi\delta(w) - \sum_{m=-\infty}^{\infty}\frac{4}{\pi(2m+1)^2}\delta(w-2\pi(2m+1))$$

$$X(w) = X_1(w) + X_2(w) = \pi\delta(w) - \sum_{m=-\infty}^{\infty}\frac{4}{\pi(2m+1)^2}\delta(w-2\pi(2m+1)) + 8\left(\frac{\sin\frac{w}{4}}{w}\right)^2 e^{-j\frac{w}{2}}$$

1.(b).

let $x(t)$ denote the function in 1.(b)

then $x(t) = x_1(t) \cdot u(t)$

where $x_1(t) =$ 

From 1.(a),

$$X_1(w) = \pi\delta(w) - \sum_{m=-\infty}^{\infty} \frac{4}{\pi(2m+1)^2}\,\delta\big(w - 2\pi(2m+1)\big)$$

We also know

$$\mathcal{F}(u(t)) = \pi\delta(w) + \frac{1}{jw}$$

$\Rightarrow$

$$X(w) = \frac{1}{2\pi} X_1(w) * \mathcal{F}(u(t))$$

$$= \frac{1}{2\pi}\left[\pi\delta(w) - \sum_{m=-\infty}^{\infty}\frac{4}{\pi(2m+1)^2}\,\delta(w-2\pi(2m+1))\right] * \left(\pi\delta(w) + \frac{1}{jw}\right)$$

$$= \frac{1}{2}\left(\pi\delta(w) + \frac{1}{jw}\right) - \sum_{m=-\infty}^{\infty}\frac{2}{\pi^2(2m+1)^2}\left[\pi\delta(w-2\pi(2m+1)) + \frac{1}{j(w-2\pi(2m+1))}\right]$$

2.(a) $f_1(t) = \sum_{k=-\infty}^{\infty} e^{j2\pi k t}$

$\Rightarrow F_1(\omega) = 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$

$\Rightarrow f_1(t) = \sum_{k=-\infty}^{\infty} \delta(t-k)$

Since $f_2(t) = \sum_{n=-\infty}^{\infty} \delta(t-n+0.5)$,

$$f_1(t) \cdot f_2(t) = 0$$

$\Rightarrow$ the output is 0

2.(b) From 2(a), $f(t) = \sum_{k=-\infty}^{\infty} e^{j2\pi k t} = \sum_{k=-\infty}^{\infty} \delta(t-k)$

$$x(t) = f(t) * h(t) = \sum_{k=-\infty}^{\infty} \sin\pi(t-k)\, \Pi(t-k-0.5)$$

$$g(t) = [x(t)]^2 = \sum_{k=-\infty}^{\infty} [\sin\pi(t-k)]^2\, \Pi(t-k-0.5)$$

$$= \sum_{k=-\infty}^{\infty} \frac{1-\cos 2\pi(t-k)}{2}\, \Pi(t-k-0.5)$$

$$= \frac{1-\cos 2\pi t}{2}$$

$$G(\omega) = 2\pi\left[\frac{1}{2}\delta(\omega) - \frac{1}{4}\delta(\omega-2\pi) - \frac{1}{4}\delta(\omega+2\pi)\right]$$

$$R(\omega) = G(\omega)\, P(\omega)$$

$$= 2\pi\left[\frac{1}{2}\delta(\omega) - \frac{1}{4}\delta(\omega-2\pi) - \frac{1}{4}\delta(\omega+2\pi)\right] e^{-|\omega|}$$

$$= 2\pi\left[\frac{1}{2}\delta(\omega) - \frac{1}{4}\delta(\omega-2\pi)\, e^{-2\pi} - \frac{1}{4}\delta(\omega+2\pi)\, e^{-2\pi}\right]$$

$$r(t) = \frac{1}{2} - \frac{1}{2} e^{-2\pi}\cos 2\pi t$$

$$= \frac{1 - e^{-2\pi}\cos 2\pi t}{2}$$

3. (a) Since the period of the sampling signal $f_2(t)$ is $3T$, the sampling frequency is $\frac{2\pi}{3T}$.

Now the maximum frequency of $f_1(t)$ is $\omega_M = 20\pi$

In order to satisfy Nyquist crieerion, we must have

$$\frac{2\pi}{3T} > 2\omega_M = 40\pi$$

$$\Rightarrow \quad T < \frac{1}{60}$$

In order to recover original signal $f_1(t)$, B must satisfy

$$\frac{2\pi}{3T} - \omega_M > B > \omega_M$$

i.e. $\quad \frac{2\pi}{3T} - 20\pi > B > 20\pi$

(b) denote $f_3(t) = f_1(t) \cdot f_2(t)$

then $F_3(\omega) = \frac{1}{2\pi} F_1(\omega) * F_2(\omega)$

$$= \frac{1}{2\pi} F_1(\omega) * \left(\sum_{n=-\infty}^{\infty} 2\pi a_k \delta(\omega - n\frac{2\pi}{3T}) \right)$$

$$= \sum_{n=-\infty}^{\infty} a_k F_1(\omega - n\frac{2\pi}{3T})$$

where a_k is the FS coefficient of $f_2(t)$.

Since $H(\omega)$ only passes $a_0 F_1(\omega)$ in order to recover $f_1(t)$,

$$A = \frac{1}{a_0}.$$

Now $a_0 = \frac{1}{3T}\int_0^{3T} f_2(t)dt = \frac{2}{3} \quad \Rightarrow A = \frac{1}{a_0} = \frac{3}{2}$

4. (a)

$$F(w_x, w_y) = 3 \cdot \frac{2\sin w_x}{w_x} \cdot \frac{2\sin \frac{3}{2} w_y}{w_y} \, e^{-3jw_x} e^{-\frac{9}{2}jw_y}$$

$$+ 2 \cdot \frac{2\sin w_x}{w_x} \cdot \frac{2\sin \frac{3}{2} w_y}{w_y} \, e^{-5jw_x} e^{-\frac{9}{2}jw_y}$$

$$+ \frac{2\sin w_x}{w_x} \cdot \frac{2\sin \frac{3}{2} w_y}{w_y} \, e^{-7jw_x} e^{-\frac{9}{2}jw_y}$$

$$= \frac{4(\sin w_x)(\sin \frac{3}{2} w_y)}{w_x w_y} e^{-\frac{9}{2}jw_y} (3e^{-3jw_x} + 2e^{-5jw_x} + e^{-7jw_x})$$

(b)

$$f_2(x, y) = f_1(-y, -x)$$

$$\Rightarrow F_2(w_x, w_y) = F_1(-w_y, -w_x)$$