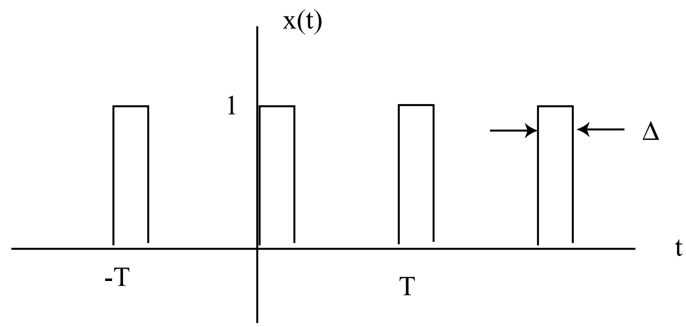


Figure 1: Periodic signal for problem 1

EECS 120. Final Exam Solution, May 19, 2000. 12.30-3.30 pm.

1. **20 points** Consider the periodic signal x of Figure 1.

   (a) Evaluate the coefficients X_k in the Fourier series

   $$\forall t, \quad x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\omega_0 t}.$$

   (b) What is ω_0? State the units.

   (c) Evaluate

   $$\sum_{k=-\infty}^{\infty} |X_k|^2.$$

Answer to 1

(a)**7** We have for $-\infty < k < \infty$,

$$X_k = \frac{1}{T}\int_0^{\Delta} 1.e^{-jk\frac{2\pi}{T}t}dt = \frac{1}{T}\frac{T}{-j2\pi k}[e^{-jk\frac{2\pi}{T}\Delta} - 1] = \frac{j}{2\pi k}[e^{-jk\frac{2\pi}{T}\Delta} - 1]$$

(b) **3** $\omega_0 = \frac{2\pi}{T}$ rad/sec.

(c) **10** By Parseval's theorem

$$\sum_{-\infty}^{\infty} |X_k|^2 = \frac{1}{T}\int_0^T |x(t)|^2 dt = \frac{\Delta}{T}.$$

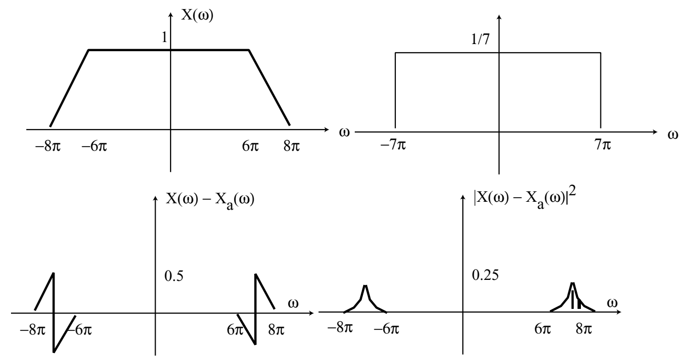


Figure 2: Signal and low pass filter for problem 2

2. **20 points** Consider the signal x with Fourier Transform given in the left of Figure 2.

   (a) What is the bandwidth of this signal in rad/sec and in Hz. What is the lowest sampling frequency in Hz that allows exact reconstruction of the signal from its samples.

   (b) Suppose x is sampled at 7 Hz and the sampled signal x_s is the product of x and $f_s(t) = \sum_{n=-\infty}^{\infty} \delta(t - n/7)$. Sketch the Fourier Transform X_s of x_s.

   (c) Let x_a be the output when x_s is passed through the low pass filter shown on the right in the figure. Sketch the Fourier transform X_a of x_a and determine the squared error $\int_{-\infty}^{\infty} |x(t) - x_a(t)|^2 dt$, without a lot of computation.

Answer to 2

(a)**5** The bandwidth is 8π rad/sec or 4 Hz. So the lowest sampling frequency is 8 Hz.

(b) The sampled signal is

$$x_s(t) = x(t) f_s(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - \frac{n}{7}),$$

So its Fourier transform is

$$\mathbf{5}\ X_s(\omega) = \frac{1}{1/7} \sum_{n=-\infty}^{\infty} X(\omega - 14n\pi) \equiv 7.$$

(c) **2** Clearly, $X_a(\omega) = 1$, for $|\omega| \leq 7\pi$, and equal to 0, otherwise. By Parseval's theorem, and the sketch of $X - X_a$ above,

$$\begin{aligned} \mathbf{4}\ \int_{-\infty}^{\infty} |x(t) - x_a(t)|^2 dt &= \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega) - X_a(\omega)|^2 |d\omega \\ \mathbf{4} \quad &= \frac{1}{2\pi} \times 4 \times \text{ shaded area} = \frac{1}{6}. \end{aligned}$$

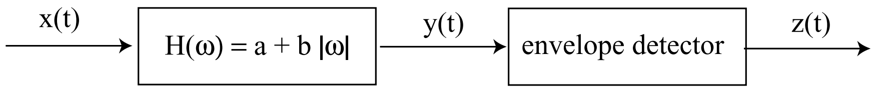


Figure 3: Demodulation scheme in Problem 3

3. **20 points** Consider a NB signal of the form

$$x(t) = \cos(\omega_c t + \theta(t)).$$

Assume $X(\omega)$ is zero except where $||\omega| - \omega_c| < 2\pi W|$ and $W << \omega_c$. Find y, z in the arrangement of Figure 3.

(Hint. In $H(\omega)$ you may use $|\omega| = (j\omega)(-j\text{sgn}\omega)$, where $\text{sgn}(w)$ is the function equal to +1 if $w \geq 0$ and -1 if $w < 0$.)

Answer to 3 Write $H(\omega) = a + b|\omega| = a + bj\omega(-j\text{sgn}(\omega))$, so

$$\begin{aligned} Y(\omega) = H(\omega)X(\omega) &= aX(\omega) + bj\omega(-j\text{sgn}(\omega))X(\omega) \\ \mathbf{2} &= aX(\omega) + bj\omega\hat{X}(\omega), \end{aligned}$$

and then, by time diffentiation property,

$$\mathbf{3}\ y(t) = ax(t) + b\frac{d}{dt}\hat{x}(t).$$

Now $x(t) = Re\{e^{j(\omega_c t + \theta(t))}\}$. The spectrum of $e^{j(\omega_c t + \theta(t))}$ is zero for negative frequencies, so

$$\mathbf{3}\ \hat{x}(t) = Im\{e^{j(\omega_c t + \theta(t))}\} = \sin(\omega_c t + \theta(t)),$$

hence

$$\mathbf{2}\ \frac{d}{dt}\hat{x}(t) = (\omega_c + \dot{\theta}(t))\cos(\omega_c t + \theta(t)),$$

so

$$\mathbf{5}\ y(t) = \{a + b(\omega_c + \dot{\theta}(t))\}\cos(\omega_c t + \theta(t)),$$

and the envelope is

$$\mathbf{5}\ z(t) = a + b(\omega_c + \dot{\theta}(t)).$$

4. **20 points** The step response of an LTI system is given by

$$s(t) = \begin{cases} 0, & t \le 0 \\ 1 - 0.5e^{-t} + 0.5e^{-2t}, & t > 0 \end{cases}$$

(a) Find its impulse response, transfer function, and frequency response.

(b) Find its steady state response to the input

$$\forall t, \quad x(t) = \cos(\omega t)u(t).$$

(c) Find its response to the input

$$\forall t, \quad x(t) = \cos(\omega t).$$

Answer to 4

(a) The Laplace transform of the step response is

$$Y(s) = \frac{1}{s} - \frac{0.5}{s+1} + \frac{0.5}{s+2} = \frac{s^2 + 2.5s + 2}{s(s+1)(s+2)}$$

so the transfer function is

$$\mathbf{3}\ H(s) = \frac{Y(s)}{1/s} = \frac{s^2 + 2.5s + 2}{(s+1)(s+2)} = 1 + \frac{0.5}{s+1} - \frac{1}{s+2},$$

the impulse response is its inverse Laplace transform,

$$\mathbf{5}\ h(t) = \delta(t) + [0.5e^{-t} - e^{-2t}]u(t),$$

and the frequency response (which exists because the system is stable) is

$$\mathbf{5}\ H(j\omega) = H(s)|_{s=j\omega} = \frac{(2 - \omega^2) + 2.5j\omega}{(2 - \omega^2) + 3j\omega}.$$

(b)**3** The steady state response is

$$|H(j\omega)| \cos(\omega t + \angle H(j\omega)).$$

(c)**4** The response is the same as the steady state response.

5. **20 points** Consider the linear vector differential equation system:

$$\begin{aligned}\dot{x}(t) &= Ax(t) + bu(t)\\ y(t) &= c'x(t) + du(t)\end{aligned}$$

where $x = (x_1, x_2)' \in R^2$ and

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad c' = [1 \quad 0] \quad d = 0.$$

(a) Find the transfer function $H(s) = Y(s)/U(s)$.

(b) The Laplace transform of the matrix-valued function $e^{tA}u(t)$ is the matrix $[sI - A]^{-1}$. Calculate this matrix and then take its inverse Laplace transform to calculate e^{tA}.

(c) Suppose the initial state is $x = (1, 1)$. Find the zero-input response for this initial state.

(d) Find the zero-state step response.

(e) Find the response when the initial state is $(1, 1)$ and the input is a unit step.

Answer to 5 We know that

$$\mathbf{3}\ H(s) = c'[sI - A]^{-1}b + d,$$

and

$$\mathbf{3}\ [sI - A]^{1} = \begin{bmatrix} s-1 & -1 \\ 0 & s-1 \end{bmatrix}^{-1} = \begin{bmatrix} (s-1)^{-1} & (s-1)^{-2} \\ 0 & (s-1)^{-1} \end{bmatrix}$$

(a) Substituting from this into H gives

$$\mathbf{3}\ H(s) = \frac{1}{(s-1)^2}.$$

(b) Taking inverse Laplace transform of $[sI - A]^{-1}$ gives

$$\mathbf{3}\ e^{tA}u(t) = \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix} u(t)$$

(c) The zero-input response due to the initial state $x_0 = (1, 1)'$ is

$$\mathbf{3}\ c'e^{tA}x_0 = [e^t + te^t]u(t).$$

(d) The Laplace transform of the zero-state step response is

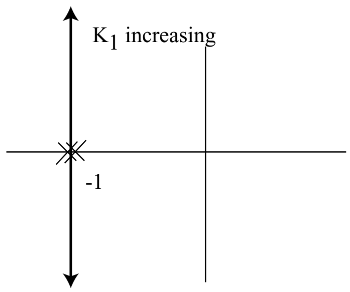
$$Y(s) = H(s)U(s) = \frac{1}{s(s-1)^2} = \frac{1}{s} - \frac{1}{s-1} + \frac{1}{(s+1)^2}$$

and taking Laplace transforms gives

3 $y(t) = [1 - e^t + te^t]u(t)$.

(e) The output is the sum of the last two, i.e.

2 $y(t) = [1 + 2te^t]u(t)$.



s^3	1	1
s^2	2	K_2
	$(2-K_2)/2$	0
	K_2	

Figure 4: Root locus in Problem 6

6. **20 points** The open loop plant has transfer function $H(s) = 1/(s^2 + 2s + 1)$. Place the plant in a closed loop using a PI controller $K_1 + K_2/s$.

   (a) Take $K_2 = 0$, and plot the root locus as K_1 varies. For what values of $K_1 \geq 0$ is the closed loop system stable? What is the steady state error to a step input as a function of K_1?

   (b) Take $K_1 = 0$, and use the Routh-Hurwitz criterion to find the values of $K_2 > 0$ such that the closed loop system is stable. What is the steady-state error for step inputs as a function of K_2?

Answer to 6

(a) **5** The root locus is shown in Figure 4.

2 The closed loop system is stable for all $K_1 > 0$. The steady-state value for step inputs is

$$\mathbf{3}\ \lim_{s\to 0} \frac{K_1 H(s)}{1 + K_1 H(s)} = \frac{K_1}{1 + K_1}$$

so the steady-state error is $1 - K_1/(1 + K_1) = 1/(1 + K_1)$.

(b) The closed loop poles are the roots of

$$\mathbf{2}\ s(s^2 + 2s + 1) + K_2 = s^3 + 2s^2 + s + K_2 = 0,$$

and for this polynomial the **(3)** Routh-Hurwitz table is given in the Figure. From the table it follows that the close-loop system is stable for **(2)** $0 < K_2 < 2$.

Finally, the steady state value to a unit step is

$$\mathbf{3}\ \lim_{s\to 0} \frac{K_2 H(s)/s}{1 + K_2 H(s)/s} = 1,$$

so the steady-state error is **(3)** zero for all values of $K_2 > 0$.

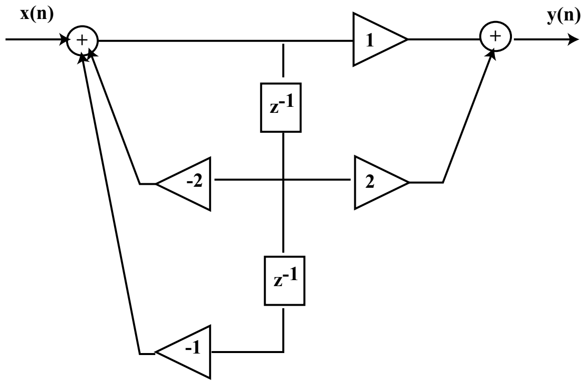


Figure 5: Direct form realization for problem 7

7. **20 points** Consider the difference equation

$$y(k) + 2y(k-1) + y(k-2) = x(k) + 2x(k-1), k = 0, 1, 2, ...$$

(a) Find the transfer function $H(z)$. Is the system BIBO stable?

(b) Find the impulse response, assuming zero initial conditions.

(c) Obtain a direct form realization of the difference equation using only two delay elements.

Answer to 7

(a) The transfer function is

$$\mathbf{2}\ H(z) = \frac{1 + 2z^{-1}}{1 + 2z^{-1} + z^{-2}} = \frac{1 + 2z^{-1}}{(1 + z^{-1})^2} = \frac{1}{(1 + z^{-1})^2} + \frac{2z^{-1}}{(1 + z^{-1})^2}.$$

Since there are two poles at -1, the system is **(3)** not BIBO stable.

(b) The impulse response is simply the inverse z transform of H,

$$\begin{aligned} \mathbf{5}\ h(n) &= (n+1)(-1)^n u(n) + 2n(-1)^{n-1} u(n-1) \\ &= 2(-1)^n u(n) - (n+1)(-1)^n u(n) \\ &= (-1)^n (1-n) u(n). \end{aligned}$$

(c)**10** The realization is given in Figure 5.

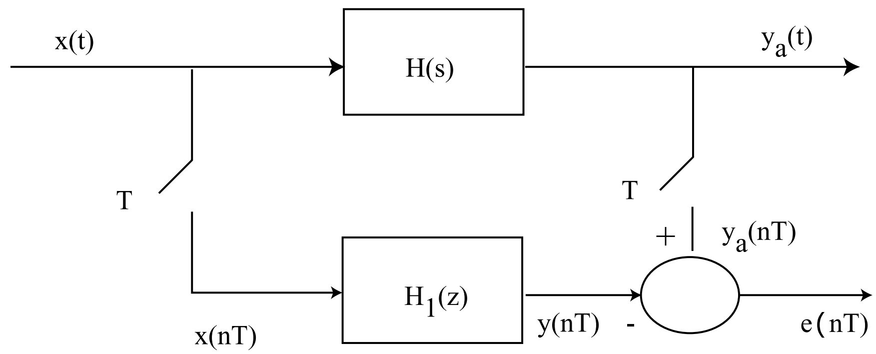


Figure 6: System for problem 8

8. **20 points**

Consider the analog transfer function $H(s) = 1/(s^2 + 5s + 6)$.

(a) Consider the scheme of Figure 6. Suppose $x(t) = u(t)$. Find $H_1(z)$ so that $e(nT) \equiv 0$, for all n. This is the step-invariant filter.

(b) Suppose $x(t) = \delta(t)$. Find $H_1(z)$ so that $e(nT) \equiv 0$. In this case assume that $x(nT)$ is the Kronecker delta. This is the impulse-invariant filter.

Answer for 8

(a) Since $x(nT) \equiv 1$,

$$X(z) = [1 - z^{-1}]^{-1}.$$

Now

$$\mathbf{3}\ Y_a(s) = \frac{1}{s(s^2 + 5s + 6)} = \frac{1/6}{s} + \frac{-1/2}{s+2} + \frac{1/3}{s+3},$$

so

$$y_a(t) = [\frac{1}{6} - \frac{1}{2}e^{-2t} + \frac{1}{3}e^{-3t}]u(t),$$

from which

$$\mathbf{3}\ y_a(nT) = [\frac{1}{6} - \frac{1}{2}e^{-2nT} + \frac{1}{3}e^{-3nT}]u(n),$$

and

$$\mathbf{3}\ Y_a(z) = \frac{1}{6}\frac{1}{1 - z^{-1}} - \frac{1}{2}\frac{1}{1 - e^{-2T}z^{-1}} + \frac{1}{3}\frac{1}{1 - e^{-3T}z^{-1}},$$

so that

$$\mathbf{3}\ H_1(z) = \frac{Y_a(z)}{X(z)} = \frac{1}{6} - \frac{1}{2}\frac{1 - z^{-1}}{1 - e^{-2T}z^{-1}} + \frac{1}{3}\frac{1 - z^{-1}}{1 - e^{-3T}z^{-1}}$$

(b) In this case $X(z) = 1$,

$$\mathbf{2}\ Y_a(s) = \frac{1}{s^2 + 5s + 6} = \frac{1}{s+2} - \frac{1}{s+3},$$

so

$$y_a(t) = [e^{-2t} - e^{-3t}]u(t),$$

from which

$$\mathbf{2}\ y_a(nT) = [e^{-2nT} - e^{-3nT}]u(n),$$

$$\mathbf{2}\ Y_a(z) = \frac{1}{1 - e^{-2T}z^{-1}} - \frac{1}{1 - e^{-3T}z^{-1}},$$

so that

$$\mathbf{2}\ H_1(z) = Y_a(z) = \frac{1}{1 - e^{-2T}z^{-1}} - \frac{1}{1 - e^{-3T}z^{-1}}.$$