# Midterm 1 Solution for EE40 Sp 2006

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1. Resistive Circuits and Capacitors
   a. This is most easily solved by using the current divider rule:
      $I_7 = I_s/8$
   b. If we treat the two resistances R in parallel as a ½R ohm resistor, we can use the voltage divider rule:
      $V_1 = V_s * ½ / (½ +1) * V_s = 1/3\ V_s$
   c. At the node of interest, we have 2I current coming in from the right, and I current leaving from the bottom, so we know that there must be I current leaving through the top. Thus, by Ohm's law, we know that $V_1 = 10+I*5$. Furthermore, we also know by Ohm's law that $V_1 = 10I$. Using these two equations, we can find that $V_1 = 20$ Volts.

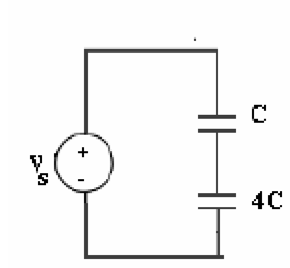
      Another way to solve this problem is to use KCL at the $V_1$ node:

      $$\frac{V_1 - 10}{5} + \frac{V_1}{10} - \frac{2V_1}{10} = 0$$

      $$2V_1 - 20 + V_1 - 2V_1 = 0$$

      $$V_1 = 20 Volts$$

   d. We know that for two capacitors in series, the charges are equal, and for two capacitors in parallel, the voltages are equal. If we treat the pair of 2C capacitors in parallel as a single 4C capacitor. Then we have the following equivalent circuit:




      We know that $Q_C=Q_{4C}$, and $V_C+V_{4C} = V_S$, $Q_C=C*V_C$, and $Q_{4C}=4C*V_{4C}$.

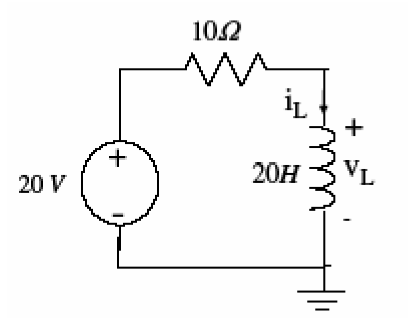
      From $Q_C = C*V_C$, $Q_{4C}=4C*V_{4C}$, and $Q_C=Q_{4C}$, we know that $V_C = 4V_{4C}$

      Thus $4V_{4C}+V_{4C}= V_S$, so $V_{4C}= 1/5 * V_S$, and $V_C = 4/5 * V_S$

      $Q_C=Q_{4C}= 4C/5 * V_S$

2. One solution is to use the trick from homework 4.

   For 0 < t < 2 sec, we have the following circuit:



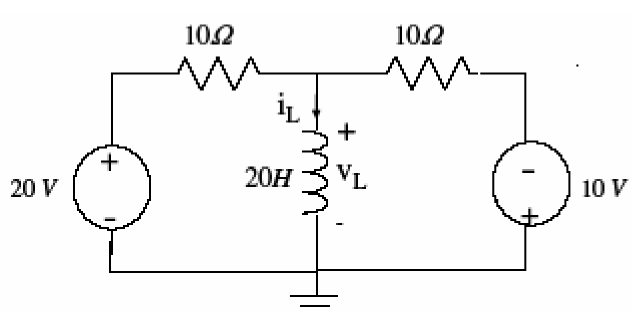

First we note that $i_L(0^-) = i_L(0^+) = 0$, and $i_L(\infty) = 20/10 = 2$ Amps

Next, we find the Thevenin resistance that the inductor sees, which is trivially 10 Ohms. This gives us the time constant L/R = 20/10 = 2 seconds

Now that we know the initial current (0 Amps), the steady state voltage (2 Amps), and the time constant (2 seconds), we use the shortcut from Homework 4 and get:

$$I_L(t) = I_f - (I_f - I_i)e^{-t/\tau}$$
$$= 2 - 2e^{-t/2\sec}\,Amps$$

For t > 2 sec, we have the following circuit:




First we note that $i_L(2^-) = i_L(2^+) = 2 - 2e^{-2/2} = 2 - 2e^{-1} = 1.26$ Amps

To find $i_L(\infty)$, we can use superposition. From the 20 volt source, $i_L(\infty)$ is increased by 2 Amps. From the 10 volt source, $i_L(\infty)$ is decreased by 1 amp. Thus, $i_L(\infty) = 2 - 1 = 1$ Amp.

Next, we find the Thevenin resistance that the inductor sees by zeroing out the independent sources, yielding the following circuit:

This is just a 10 ohm resistor in parallel with another 10 ohm resistor, which means that the resistance the inductor sees is 5 ohms.

Thus our time constant is 20/5=4 seconds.

So, again using the trick from homework 4, we have

$$I_L(t) = I_f - (I_f - I_i)e^{-(t-2)/\tau}$$
$$= 1 - (1 - 1.26)e^{-(t-2)/4}$$
$$= 1 + 0.26e^{-(t-2\sec)/4\sec} Amps$$

Another possible solution is to write the differential equations in both cases and solve.

For the first case, we can use KVL to find that:

$$10 - 10I_L - 20I_L^{'} = 0$$
$$2 - I_L - 2I_L^{'} = 0$$
$$I_L + 2I_L^{'} = 2$$

We first find the complementary solution $I_C(t) = Ke^{-t/\tau}$. Since we have our equation in the form $I_L + \tau I_L^{'} = f(t)$, we know that $I_C(t) = Ke^{-t/2\sec} Amps$.

Next we can find the particular solution by guessing that our solution is of the form $I_P(t) = A * f(t) + B * f'(t) = A$. Plugging this into our differential equation above, we get that A+0=2, or A=2.

Finally, we know that I(0)=0, so we find I(0)= $I_C(0) + I_P(0) = K + 2 = 0$, or K=-2. Thus, our final solution for 0 < t < 2 is $I(t) = 2 - 2e^{-t/2\sec} Amps$.

For the second part of the problem, we write a new differential equation using KCL at our node of interest. (We could also write two KVL equations and add them).

Doing this, we obtain:

$$\frac{V_L - 20}{10} + \frac{V_L + 10}{10} + \int \frac{V_L}{20} = 0$$
$$V_L = LI_L^{'}$$
$$\frac{20I_L^{'} - 20}{10} + \frac{20I_L^{'} + 10}{10} + \frac{20I_L}{20} = 0$$
$$2I_L^{'} - 2 + 2I_L^{'} + 1 + I_L = 0$$
$$4I_L^{'} - 1 + I_L = 0$$
$$4I_L^{'} + I_L = 1$$

Since our equation is in the form $I_L + \tau I_L' = f(t)$, we know that our complementary solution is of the form $I_{L_C}(t) = Ke^{-(t-2\sec)/4\sec} Amps$.

Next we find the particular solution. As above, we assume that $I_{L_P}(t) = A$, and plug this into our differential equation, finding that A=1.

Now we add our particular and complementary solution and have that $I_L = Ke^{-(t-2)/4} + 1$. To find K, we know that $I_L(2) = 2 - 2e^{-1} = 1.26A$, so $I_L(2) = Ke^{-(2-2)/4} + 1 = K + 1 = 1.26A$, and therefore K=0.26A.

Thus our final solution is $I_L(t) = 1 + 0.26e^{-(t-2\sec)/4\sec} Amps$

There are many more possible solutions, such as using separation of variables, etc,

3.

a. For t < 0, the circuit has been closed for a long time, and since we have a DC source, we can perform DC steady state analysis. We treat the capacitor as an open circuit, and the inductor like a short. Thus, we find that $I_L = 5/500{,}000 = 10^{-5} Amps$, and since the inductor acts like a short, $V_C = 0$ Volts.

b. One method is to write KCL at the node, take the derivative of both sides, and reorganize the terms, as shown below:

$$\frac{V_L - \cos(t)}{R} + CV_L' + \int \frac{V_L}{L} = 0$$

$$\frac{V_L' + \sin(t)}{R} + CV_L'' + \frac{V_L}{L} = 0$$

$$\frac{V_L'}{RC} + V_L'' + \frac{V_L}{LC} = -\frac{\sin(t)}{RC}$$

$$V_L'' + 2V_L' + V_L = -2\sin(t)$$

c. Since, our equation is in the form

$$\frac{d^2x(t)}{dt^2} + 2\alpha\frac{dx(t)}{dt} + \omega_0^2 x(t) = f(t)$$

, we know that

$$\alpha = 1,\ \omega_0 = 1,\ \zeta = \frac{\alpha}{\omega_0} = \frac{1}{1} = 1$$

Therefore, the complementary/transient solution is:

$$V_C(t) = K_1 e^{-\alpha t} + K_2 t e^{-\alpha t} \text{ Volts}$$

d. Critically damped

e. We assume a solution of the form $V_p(t) = A\cos(t) + B\sin(t)$

$$V_p(t) = A\cos(t) + B\sin(t)$$

$$V_p^{'}(t) = -A\sin(t) + B\cos(t)$$

$$V_p^{''}(t) = -A\cos(t) - B\sin(t)$$

Then we plug them into our equation from part b, and get:

$$-A\cos(t) - B\sin(t) - 2A\sin(t) + 2B\cos(t) + A\cos(t) + B\sin(t) = -2\sin(t)$$

By collecting sine and cosine terms, we find the following:
-A+2B+A=0
-B-2A+B = -2

From the first equation we find that B=0.

Plugging B=0 into the second equation, we get -2A=-2, or A=1.

Thus our particular solution $V_p(t) = \cos(t)$

f. We obtain the complete solution by adding the particular solution and the complementary solution, so we have:

$$V(t) = K_1 e^{-t} + K_2 te^{-t} + \cos(t)$$

To find the constants, we can first use the initial condition v(0)=0, and obtain:

$$V(0) = K_1 e^0 + K_2 0e^0 + \cos(0) = 0$$

$$K_1 + 1 = 0$$

$$K_1 = -1$$

To find $K_2$, we know that $i_L(0^-)= i_L(0^+)=10^{-5}$ *Amps*. However, to use this information directly, we'd need an equation for $I_L$(t).

Instead, it's easier to find $i_C(0^+)$. Note: $i_C(0^+) \neq i_C(0^-)$!! At time $0^+$, we can write KCL at node $V_L$, which gives us:

$$\frac{V_L(0^+) - \cos(0)}{500000} + I_C(0^+) + I_L(0^+) = 0$$

We also know the following facts:

$V_L(0^+) = V_C(0^+) = V_C(0^-) = 0$
$I_L(0^+) = I_L(0^-) = 10^{-5}\ Amps$

So our above KCL equation becomes:

$$\frac{-10^{-5}}{5} + I_C(0^+) + 10^{-5}\ Amps = 0$$

This gives us:
$I_C(0^+) = -4/5 * 10^{-5}$ amps

Next, we find $I_C(t) = CV_C'(t) = 10^{-6}(e^{-t} + K_2 e^{-t} - K_2 t e^{-t} - \sin(t))$, and thus :

$I_C(0) = 10^{-6}(1 + K_2) = -4/5 * 10^{-5}$
$(1 + K_2) = -40/5$
$(1 + K_2) = -8$
$K_2 = -9$

Thus, we have our final solution:

$$V_L(t) = -e^{-t/\sec} - 9te^{-t/\sec} + \cos(t)\ \text{Volts}$$