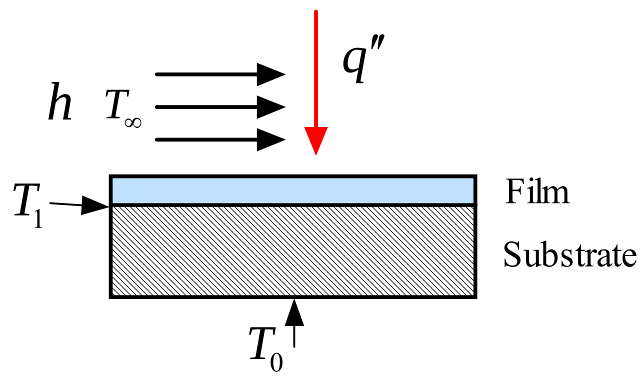
ME109 – Heat Transfer
Midterm 1- Fall'06
Instructor: Prof. A. Majumdar
Oct. 13, 2006; 12:10 am - 1:00 pm; Maximum Points = 30

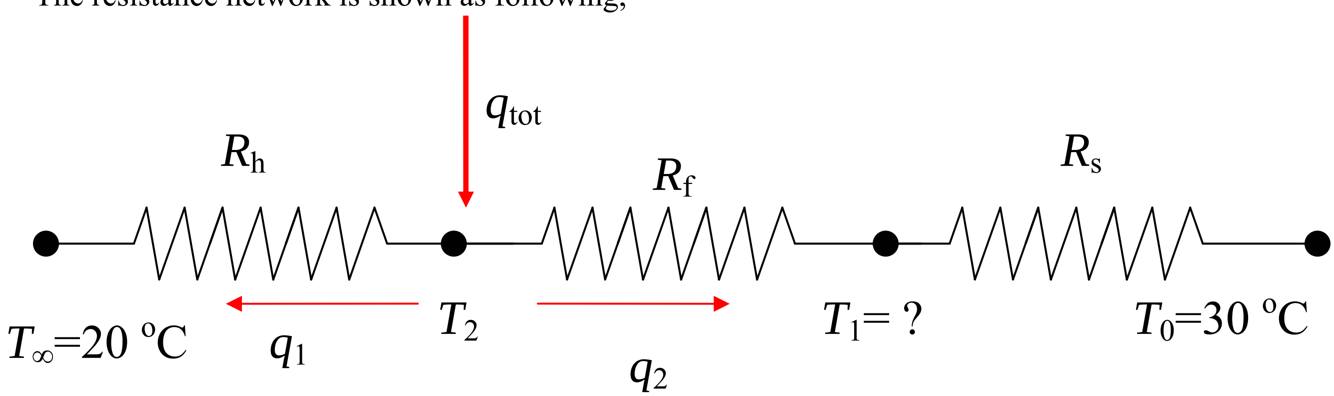
Suggested Solutions:

Problem 1:



This is a steady-state, one-dimensional heat transfer problem, *without* heat generation, so we can we the thermal resistance network.

The resistance network is shown as following,



Conservation of energy requires,

$$q_1 + q_2 = q_{tot} = q''A \qquad (1)$$

where A is the cross-sectional area of both the film and the substrate.

According to the Fourier Law:

$$q_1 = \frac{T_2 - T_\infty}{R_h} \qquad (2)$$

$$q_2 = \frac{T_2 - T_0}{R_f + R_s} \tag{3}$$

where

$$\begin{aligned} R_h &= \frac{1}{hA} = \frac{1}{40A} = \frac{0.025}{A}\left[K/W\right] \\ R_f &= \frac{a}{k_f A} = \frac{0.0002}{0.02A} = \frac{0.010}{A}\left[K/W\right] \\ R_s &= \frac{b}{k_s A} = \frac{0.001}{0.06A} = \frac{0.0167}{A}\left[K/W\right] \end{aligned} \tag{4}$$

Substitute Eq.(2) and Eq.(3) into Eq.(1):

$$\frac{T_2 - T_\infty}{R_h} + \frac{T_2 - T_0}{R_f + R_s} = q''A \tag{5}$$

From Eq.(5), solve for T_2

$$T_2 = \frac{q''A + T_\infty/R_h + T_o/\left(R_s + R_f\right)}{1/R_h + 1/\left(R_s + R_f\right)} \tag{6}$$

$$T_2 = 63.55[^oC]$$

Then substitute T_2 into Eq.(3),

$$q_2 = \frac{T_2 - T_0}{R_f + R_s} = \frac{63.55 - 30}{0.0167 + 0.01}A \tag{7}$$

$$q_2 = 1258A[W]$$

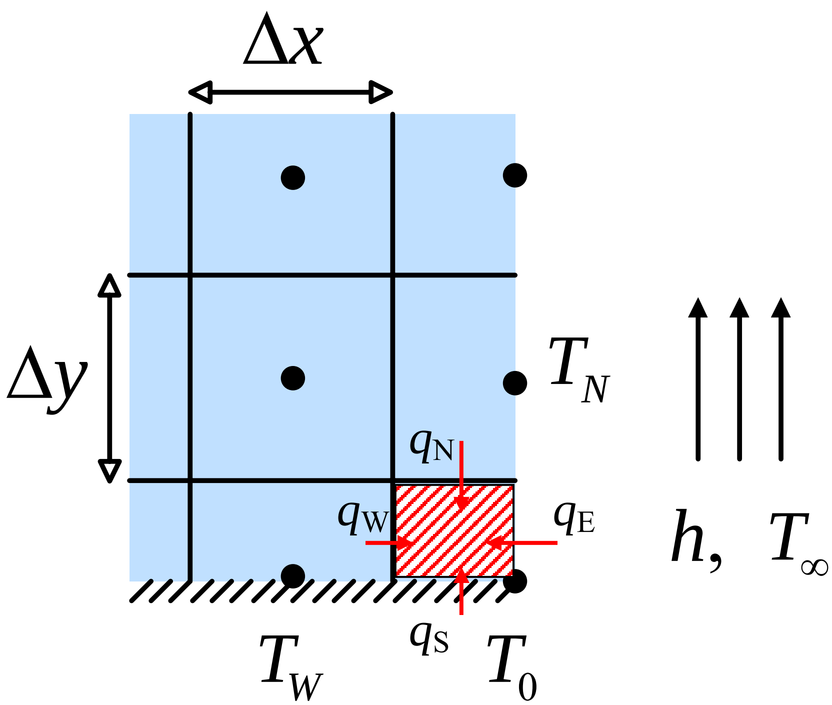
Apply the Fourier law on R_s

$$T_1 - T_0 = q_2 R_s \tag{8}$$

$$T_1 = T_0 + q_2 R_s = 30 + 1258A\frac{0.0167}{A} \cong 51[^oC]$$

Therefore, the temperature at the film/substrate interface T_1 is 51 [^{o}C]

Problem 2:



Set up the control volume (C.V.) for node T_0, as shown in the above figure.
Use the 1st law; write down the energy conservation equation for the C.V.:

$$\frac{dE_{CV}}{dt} = \dot{E}_{in} - \dot{E}_{out} + \dot{E}_g \tag{1}$$

where

$$\frac{dE_{CV}}{dt} = \rho c V \frac{dT}{dt} \tag{2}$$

$$\dot{E}_{in} = q_N + q_W + q_S + q_E = kA_N \frac{\partial T}{\partial y} - kA_W \frac{\partial T}{\partial x} + 0 + hA_E \left(T_\infty - T_o\right) \tag{3}$$

$$\dot{E}_{out} = 0 \tag{4}$$

$$\dot{E}_g = q''' V \tag{5}$$

Express each term in the discrete form, and notice that the explicit scheme is required.

$$\frac{dE_{CV}}{dt} = \rho c \frac{\Delta x \Delta y}{4} \frac{dT}{dt} = \rho c \frac{\Delta x \Delta y}{4} \frac{T_0^{p+1} - T_0^p}{\Delta t} \tag{6}$$

$$\dot{E}_{in} = q_N + q_W + q_S + q_E = k\frac{\Delta x}{2}\frac{T_N{}^p - T_o{}^p}{\Delta y} + k\frac{\Delta y}{2}\frac{T_N{}^p - T_o{}^p}{\Delta x} + 0 + h\frac{\Delta y}{2}\left(T_\infty - T_o{}^p\right) \tag{7}$$

$$\dot{E}_g = q''' \frac{\Delta x}{2}\frac{\Delta y}{2} = q''' \frac{\Delta x \Delta y}{4} \tag{8}$$

Substitute Eq.(6)-(8) into Eq.(1), we obtain:

$$. \rho c \frac{\Delta x \Delta y}{4} \frac{T_0^{p+1} - T_0^p}{\Delta t} = k \frac{\Delta x}{2} \frac{T_N{}^p - T_o{}^p}{\Delta y} + k \frac{\Delta y}{2} \frac{T_N{}^p - T_o{}^p}{\Delta x} + 0 + h \frac{\Delta y}{2} \left(T_\infty - T_o{}^p \right) + q''' \frac{\Delta x \Delta y}{4} \quad (9)$$

Note $\Delta x = \Delta y$, Eq.(9) can be simplified to:

$$T_0^{p+1} = T_o{}^p \left(1 - 4 \frac{k \Delta t}{\rho c \Delta x^2} - 2 \frac{h \Delta t}{\rho c \Delta x} \right) + 2 \frac{k \Delta t}{\rho c \Delta x^2} \left(T_N{}^p + T_W{}^p \right) + 2 \frac{h \Delta t}{\rho c \Delta x} T_\infty + \frac{\Delta t}{\rho c} q''' \qquad (10a)$$

Define:

$$F_o \equiv \frac{\alpha \Delta t}{\Delta x^2} = \frac{k \Delta t}{\rho c \Delta x^2}$$

$$Bi \equiv \frac{h \Delta x}{k}$$

Eq.(10a) can be written as:

$$T_0^{p+1} = T_o{}^p \left(1 - 4F_o - 2F_o B_i \right) + 2F_o \left(T_N{}^p + T_W{}^p \right) + 2F_o B_i T_\infty + \frac{\Delta t}{\rho c} q''' \qquad (10b)$$

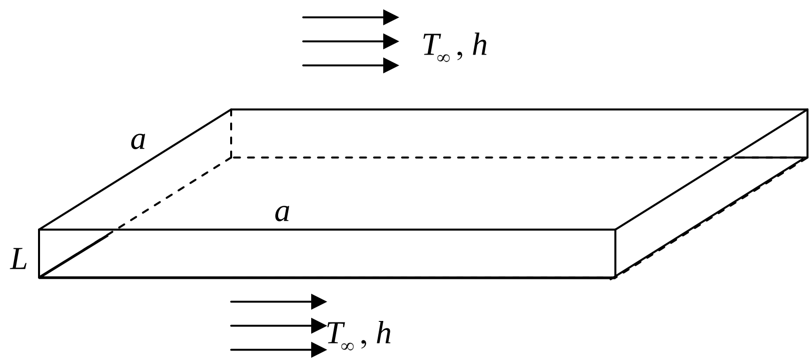
The stability criterion requires that:

$$\left(1 - 4F_o - 2F_o B_i \right) \geq 0 \qquad (11a)$$

or

$$F_o \left(2 + B_i \right) \leq \frac{1}{2} \qquad (11b)$$

Problem 3:



To justify the validity of the lumped capacitance method, we calculate the Bi numbers at x, y and z direction:

$$Bi_x = Bi_y = \frac{ha}{k} = \frac{30 \times 0.6}{400} = 0.045 < 0.1$$

$$Bi_z = \frac{hL}{k} = \frac{30 \times 0.02}{400} = 0.015 < 0.1$$

The Bi numbers at the all the direction are smaller than 0.1, hence the validity of the lumped capacitance method is justified.

Let the whole link be the control volume, which is at the uniform temperate $T(t)$. We can do so because the system is treated as a lumped one as we have shown above.

The conservation of energy (1st law) requires:

$$\frac{dE_{CV}}{dt} = \dot{E}_{in} - \dot{E}_{out} + \dot{E}_g \tag{1a}$$

or

$$\rho c V \frac{dT}{dt} = hA\left(T_\infty - T\right) \tag{1b}$$

Define:

$$\theta \equiv T - T_\infty \tag{2}$$

Eq.(1) can be written as:

$$\rho c V \frac{d\theta}{dt} = -hA\theta \tag{3a}$$

or

$$\frac{d\theta}{dt} = -\theta/\tau \tag{3b}$$

where

$$\tau = \frac{\rho c V}{hA} \tag{4}$$

The initial condition is:

$$t = 0,\ T = T_i$$

or

$$t = 0,\ \theta = \theta_i \equiv T - T_i \tag{5}$$

Integrate Eq.(3b) and apply the initial condition Eq.(5):

$$\theta = \theta_i \exp\left(-\frac{t}{\tau}\right) \tag{6}$$

In this problem,

$$\tau = \frac{\rho c V}{hA} = \frac{\rho c \times La^2}{h \times 2a^2} = \frac{\rho c L}{2h} = 1000[s] \tag{7}$$

(Note: if consider A = $a^2 \cdot 2 + 4aL$=0.768 [m^2], $\tau = 937.5[s]$, which is *also* considered to be correct.)

Solve for the time for the temperature reaching 297 [^{o}C]:

$$\frac{\theta}{\theta_i} = \frac{T - T_\infty}{T - T_\infty} = \exp\left(-\frac{t}{\tau}\right)$$

$$t = -\tau \ln\left(\frac{T - T_\infty}{T - T_\infty}\right) \tag{8}$$

$$t = -\tau \ln\left(\frac{297 - 300}{20 - 300}\right)$$

$$t = 4.536\ \tau$$

$$\text{t} = 4536\ [\text{s}]$$

(If using $\tau = 937.5[s]$, then t = 4253 [s]).