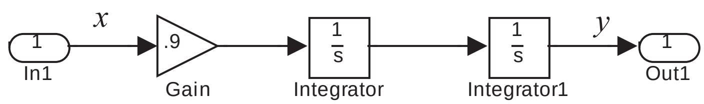
EECS 20. Midterm No. 2 Solutions
November 8, 2002.

1. **40 points (10 points each part).** Consider the Simulink diagram shown below:



This shows an LTI system with one input and one output, both of which are continuous-time signals. The input and output are indicated by the rounded boxes, and are labeled x and y. The gain is 0.9, and the integrators both have initial condition equal to 0.

(a) Write a differential equation (with no integrals, just derivatives) that relates the input x and the output y.
Answer:

$$\forall\, t \in \textit{Reals}_+, \quad 0.9x(t) = \ddot{y}(t).$$

(b) Give the $[A, b, c, d]$ representation of this system.
Solution: Let the state be the outputs of the two integrators, as follows:

$$\forall\, t \in \textit{Reals}_+, \quad z(t) = \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix}$$

Hence,

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ 0.9 \end{bmatrix} \quad c^T = [1\ \ 0] \quad d = 0.$$

(c) Find the frequency response $H\colon \textit{Reals} \to \textit{Reals}$ of this system.
Solution: Let the input be $x(t) = e^{i\omega t}$, in which case the output must be $H(\omega)e^{i\omega t}$. Substituting these into the differential equation from part (a), we get

$$0.9e^{i\omega t} = H(\omega)(-\omega^2 e^{i\omega t}),$$

which we can solve for $H(\omega)$ to get

$$H(\omega) = -0.9/\omega^2.$$

(d) Find the output of the system if the input x is given by

$$\forall\, t \in \textit{Reals}, \quad x(t) = \cos(2t).$$

Solution: Write this input as

$$\forall\, t \in \textit{Reals}, \quad x(t) = 0.5(e^{i2t} + e^{-i2t}),$$

which using linearity implies that the output must be

$$\forall\, t \in \textit{Reals}, \quad y(t) = 0.5(H(2)e^{i2t} + H(-2)e^{-i2t}) = 0.5((0.9/4)e^{i2t} + (0.9/4)e^{-i2t}) = 0.2\cos(2t).$$

2. **30 points (5 points each part).** Consider continuous-time systems with input $x\colon \textit{Reals} \to \textit{Reals}$ and output $y\colon \textit{Reals} \to \textit{Reals}$. Each of the following defines such a system. For each, indicate whether it is linear (L), time-invariant (TI), both (LTI), or neither (N). Note that no partial credit will be given for these questions.

   (a) $\forall\, t \in \textit{Reals}, \quad \dot{y}(t) = x(t) + 0.9y(t)$ **Answer**: LTI

   (b) $\forall\, t \in \textit{Reals}, \quad y(t) = \cos(2\pi t)x(t)$ **Answer**: L

   (c) $\forall\, t \in \textit{Reals}, \quad y(t) = x(t-1)$ **Answer**: LTI

   (d) $\forall\, t \in \textit{Reals}, \quad y(t) = x(t) + 0.1(x(t))^2$ **Answer**: TI

   (e) $\forall\, t \in \textit{Reals}, \quad y(t) = x(t) + 0.1(x(t-1))^2$ **Answer**: TI

   (f) $\forall\, t \in \textit{Reals}, \quad y(t) = 0$ **Answer**: LTI

3. **40 points (10 points each part).** Consider a discrete-time signal $x\colon \textit{Integers} \to \textit{Reals}$ defined by

$$\forall\, n \in \textit{Integers}, \quad x(n) = 1 - \cos(3\pi n/4).$$

   Assume this signal is sampled at 8,000 samples/second.

   (a) Give the frequency of the cosine term in Hz (cycles/second).

   Solution: The frequency is $3\pi/4$ radians/sample. Divide by 2π radians/cycle and multiply by 8000 samples/second to get 3000 Hz.

   (b) Give period of x.

   Solution: The period is the smallest positive integer p such that $3\pi p/4$ is a multiple of 2π. Thus, $p = 8$.

   (c) Give the fundamental frequency (in any units, but be sure to give the units).

   Solution: $\omega_0 = 2\pi/p = \pi/4$ radians/sample. Alternatives: 1/8 cycles/sample, or 8000/8 cycles/second = 1000 cycles/second.

   (d) Give the coefficients $A_0, A_1, A_2, \cdots, A_K$ and $\phi_1, \phi_2, \cdots, \phi_K$ of the Fourier series expansion for x,

$$x(n) = A_0 + \sum_{k=1}^{K} A_k \cos(k\omega_0 n + \phi_k)$$

   where

$$K = \begin{cases} (p-1)/2 & \text{if } p \text{ is odd} \\ p/2 & \text{if } p \text{ is even} \end{cases}$$

   Solution: $A_0 = 1$, $A_1 = 0$, $A_2 = 0$, $A_3 = 1$, $A_4 = 0$, and $\phi_1 = 0$, $\phi_2 = 0$, $\phi_3 = \pi$, $\phi_4 = 0$, although the phases corresponding to zero amplitude can be anything. The $\phi_3 = \pi$ accounts for the minus sign in the definition of x.