EECS 20. Solutions to Midterm 1. 2 October 1998

1. **15 points**

   (a) Find θ so that
   $$Re[(1+i)\exp i\theta] = -1.$$

   Answer Using $\exp(i\theta) = \cos(\theta) + i\sin(\theta)$,
   $$Re[(1+i)\exp i\theta = Re[\cos(\theta) - \sin(\theta) + i(\cos(\theta) + \sin(\theta)] = -1$$
   so
   $$\cos(\theta) - \sin(\theta) = -1,$$
   one solution of which is $\theta = \pi/2$. Another solution is $\theta = \pi$. The general solution is $\pi/2 \pm 2n\pi, \pi \pm 2n\pi$.

   (b) Define $x : Reals \to Reals$
   $$\forall t \in Reals, x(t) = \sin(\omega_0 t + 1/4\pi).$$
   Find $A \in Comps$ so that
   $$\forall t \in Reals, x(t) = A\exp(i\omega_0 t) + A^* \exp(-i\omega_0 t),$$
   where A^* is the complex conjugate of A.
   Answer Using $\sin(\theta) = 1/2i[\exp(i\theta) - \exp(-i\theta)]$,
   $$\sin(\omega_0 t + 1/4\pi) = 1/2i[\exp(i(\omega_0 t + 1/4\pi)) - \exp(-i(\omega_0 t + 1/4\pi))$$
   so $A = 1/2i\exp(i1/4\pi) = 1/2[\sin(\pi/4) - i\cos(\pi/4)]$.

2. **15 points**
Draw the following sets

(a) $\{(x, y) \in Reals^2 \mid xy = 1\}$.

(b) $\{(x, y) \in Reals^2 \mid y - x^2 \geq 0\}$.

(c) $\{z \in Comps \mid z^5 = 1 + 0i\}$.

Answer In drawing the third set, we use the fact that $z^5 = 1 = \exp i(2n\pi)$, so $z = \exp i(2n\pi/5)$, $n = 0, 1, 2, 3, 4$. There are no more solutions, since $\exp i(2 \times 5\pi/5) = expi2\pi = 1$ which we have already.

Note A very important result of complex variables is that a polynomial of degree n in a complex variable z has exactly n roots. In the case here check that

$$z^5 - 1 = \prod_{n=0}^{4} (z - \exp i(2n\pi/5)).$$

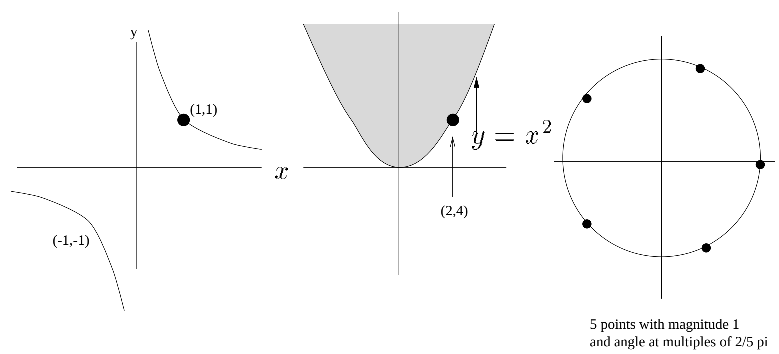

Figure 1: The three sets

3. **25 points**

(a) Evaluate the truth values of

$$S = [P \wedge (\neg Q)] \vee R$$

for the following values of P, Q, R.

P	Q	R	S
True	False	False	
False	True	False	
True	False	True	

Answer

P	Q	R	S
True	False	False	**True**
False	True	False	**False**
True	False	True	**True**

(b) The following sequence of statements is a complete context.

Let

$$x = 5, y = 6 \quad (1)$$

Then,

$$x \neq y \quad (2)$$

Now let

$$Z = \{z \in \mathit{Reals} \mid z \geq x + y\} \quad (3)$$

Then

$$x \in Z \quad (4)$$

Let

$$w = \text{ smallest non-negative number in } Z \quad (5)$$

Answer the following:

i. Are the two expressions in (1) both assignments or assertions?
Ans Both are assignments

ii. Is the expression (2) an assertion or a predicate?
Ans It is a true assertion

iii. Is the equality in (3) an assignment or an assertion?
Ans It is an assignment in which Z is assigned the set on the right-hand side.

iv. Is the expression "$z \geq x + y$" in (3) an assertion or a predicate?
Ans It is a predicate which is true if and only if $z \geq 11$

v. Is (4) an assertion or a predicate?
Ans It is a false assertion

vi. Is (5) an assignment or an assertion?
Ans It is an assignment equivalent to the assignment $w = 11$.

4. **20 points**

A signal is a function. We have studied signals that are functions of time and space and functions that are data and event sequences. Mathematically, we model a signal as a function with some range and common domain. For example, $Sound : Time \to Pressure$. Propose mathematical models for the signals with the following intuitive descriptions. Give a very brief justification for your proposed models.

(a) A gray-scale video with 256 gray-scale values .
Ans A video is a sequence of images. Let

$$Images = [HorSpace \times VerSpace \to \{0, \cdots, 255\}]$$

Then a video is represented by a function

$$Video : Time \to Images$$

where $Time = \{0, 1/30, 2/30, \cdots\}$.

(b) The position of a bird in flight.
Ans The bird's position at time t in flight can be represented as a point in three-dimensional space, $(x(t), y(t), z(t))$, so

$$Position : Time \to Reals^3$$

where $Time = [a, b]$ is the duration of the flight.

(c) The buttons you press with your TV remote control.
Ans Let $Buttons = \{power, play, fwd, rew, \cdots\}$ be the buttons we can press. Then the sequence of button presses can be modeled by a function

$$ButtonPress : Indices \to Buttons$$

where $Indices = \{1, 2, \cdots\}$

5. **25 points** The function $x : Reals \to Reals$ is given by its graph shown in Figure 2. Note that $\forall t \notin [0,1], x(t) = 0$, and $x(0.4) = 1$. Define y by

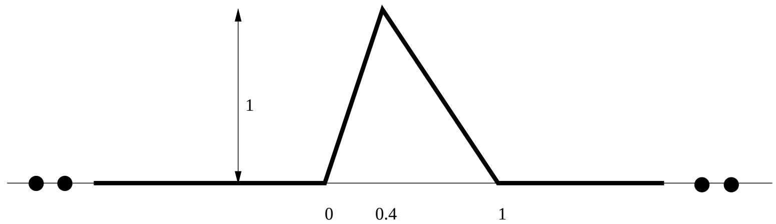

Figure 2: The graph of x

$$\forall t \in Reals, y(t) = \sum_{k=-\infty}^{\infty} x(t - kp)$$

where $p \in Reals$.

(a) Prove that y is periodic with period p, i.e.

$$\forall t \in Reals, \;\; y(t) = y(t+p).$$

Ans We must verify this using the definition of y. Substituting $t+p$ for t we get

$$\begin{aligned} y(t+p) &= \sum_{k=-\infty}^{\infty} x(t+p-kp) \\ &= \sum_{k=-\infty}^{\infty} x(t+(1-k)p) \\ &= \sum_{m=-\infty}^{\infty} x(t-mp), \text{ by taking } m = 1-k \\ &= y(t), \text{ by definition of } y \end{aligned}$$

(b) Plot y for $p = 1$.

(c) Plot y for $p = 2$.

(d) Plot y for $p = 0.5$.

Ans See Figure 3. Note that the period in the top plot is 1.0, in the middle it is 2.0 and in the lower plot it is 0.5.

(e) Suppose the function z is obtained by advancing x by 0.4, i.e.

$$\forall t, \;\; z(t) = x(t + 0.4).$$

Define w by

$$\forall t \in Reals, w(t) = \sum_{k=-\infty}^{\infty} z(t - kp)$$

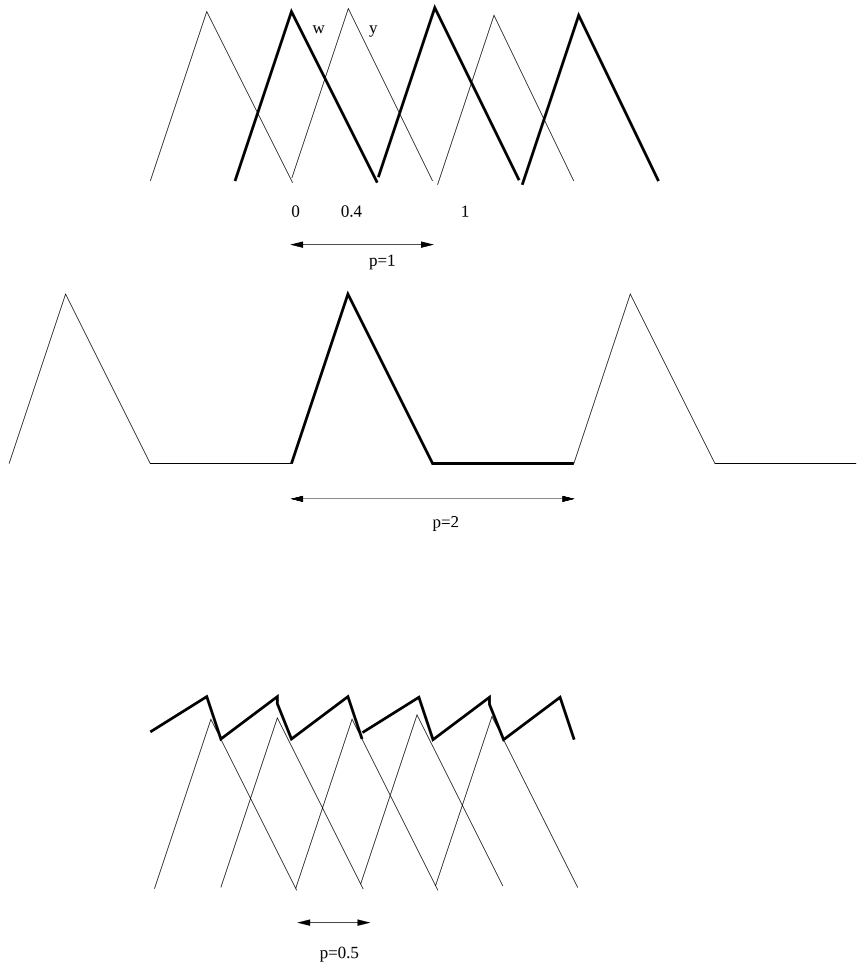

Figure 3: The graphs of y, w

What is the relation between w and y. Use this relation to plot w for $p = 1$.
Ans We have

$$
\begin{aligned}
w(t) &= \sum_{k=-\infty}^{\infty} z(t - kp) \\
&= \sum_{k=-\infty}^{\infty} x(t + 0.4 - kp) \\
&= y(t + 0.4)
\end{aligned}
$$

So the plot of w is obtained by moving the plot of y to the left by 0.4, as shown in the top panel of Figure 3 in bold.