EECS20n, Solution to Midterm 1, 10/20/00

1. **10 points** The function $x : Reals \to Reals$ given by

$$\forall t \in Reals \quad x(t) = \sin(2\pi \times 440t)$$

is a mathematical example of a signal in the signal space $[Reals \to Reals]$. Give mathematical examples of signals in the following signal spaces.

(a) **(2 pts)** $[Ints \to Reals]$

$$\forall n \in Ints, \quad x(n) = n$$

(b) **(2 pts)** $[Nats_0 \to EnglishWords]$

$$\forall n \in Nats_0, \quad x(n) = the$$

(c) **(2 pts)** $[Reals \to Reals^2]$

$$\forall t \in Reals, \quad x(t) = (t, 2t)$$

(d) **(2 pts)** $[\{0, 1, \cdots, 600\} \times \{0, 1, \cdots, 400\} \to \{0, 1, \cdots, 255\}]$

$$\forall (m, n) \in \{0, 1, \cdots, 600\} \times \{0, 1, \cdots, 400\}, \quad x(m, n) = (m + n) \bmod 255$$

(e) **(2 pts)** Give an example of a practical space of signals whose mathematical representation is $[\{0, 1, \cdots, 600\} \times \{0, 1, \cdots, 400\} \to \{0, 1, \cdots, 255\}]$.

This is the appropriate signal space for images of size 600×400 pixels with an 8-bit color map index.

2. **10 points** The function $H : [\textit{Reals}_+ \to \textit{Reals}] \to [\textit{Nats}_0 \to \textit{Reals}]$ given by: $\forall x \in [\textit{Reals}_+ \to \textit{Reals}]$,

$$\forall n \in \textit{Nats}_0, \quad H(x)(n) = x(10n),$$

is a mathematical example of a system with input signal space $[\textit{Reals}_+ \to \textit{Reals}]$ and output signal space $[\textit{Nats}_0 \to \textit{Reals}]$. Give mathematical examples of systems whose

(a) **5 pts** input and output signal spaces both are $[\textit{Nats}_0 \to \textit{Bin}]$.

The simplest example is the identity system:

$$\forall x, \forall n, \quad H(x)(n) = x(n).$$

(b) **5 pts** input signal space is $[\textit{Nats}_0 \to \textit{Reals}]$ and output signal space is $[\textit{Nats}_0 \to \{0, 1\}]$.

A simple example is a system which converts negative values to 0 and positive values to 1:

$$\forall x, \forall n, \quad H(x)(n) = \begin{cases} 0, & \text{if } x(n) \leq 0 \\ 1, & \text{if } x(n) > 0 \end{cases}$$

(c) **5 pts** input signal space is $[\textit{Ints} \to \textit{Reals}]$ and output signal space is $[\textit{Reals} \to \textit{Reals}]$.

A simple example is a “zero-order” hold:

$$\forall x, \forall t \in \textit{Reals}, \quad H(x)(t) = x(n), \text{ where } n = \lfloor t \rfloor.$$

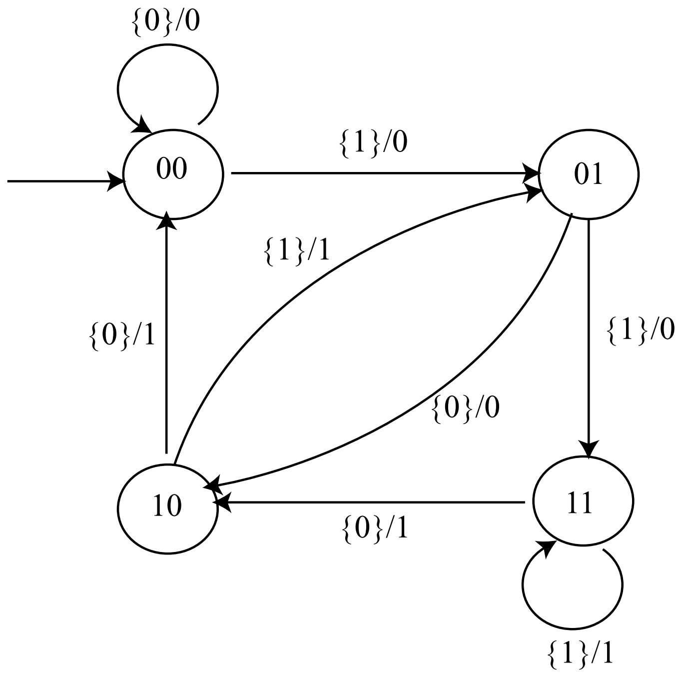

Figure 1: Machine for problem 3b

3. **10 points** A state machine has *Inputs* $=$ *Outputs* $= \{0, 1\}$.

   (a) **3 pts**What is the space of input signals and the space of output signals of this state machine?

   $$InputSignals = OutputSignals = [Nats_0 \to \{0, 1\}].$$

   (b) **5 pts** Construct a *deterministic* machine whose input-output function H is given by (letting x denote the input signal and $y = H(x)$ denote the output signal): $\forall n \geq 0$,

   $$y(n) = \begin{cases} 0, & \text{if } n = 0, 1 \\ x(n-2), & \text{if } n \geq 2 \end{cases}$$

   This is a delay-2 machine. The state must remember the two previous inputs. So there are 4 states denoted ij where i is the input two steps before and j is the previous input. The machine is shown in figure 1. *initialState* $= 00$ since the first two ouputs are 0.

   (c) **2 pts** What is the output of your machine when the input is $0, 1, 0, 1, \cdots$?

   The output is $0, 0, 0, 1, \cdots$

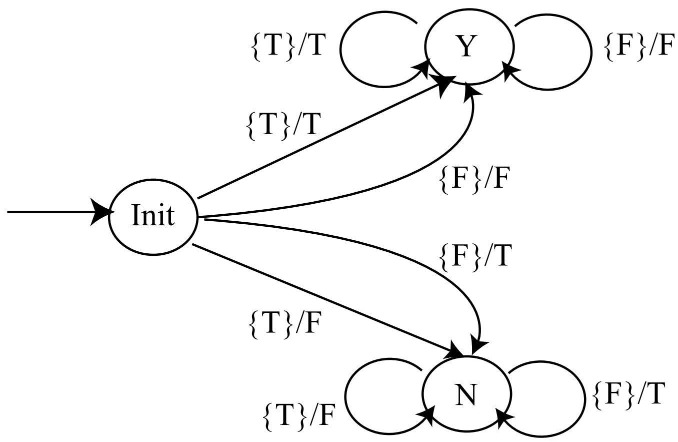

Figure 2: The machine for problem 4

4. **10 points** Construct a *non-deterministic* state machine with *Inputs* = *Outputs* = $\{T, F\}$ which for any input signal x has two possible output signals y, namely $y = x$, and $y = \bar{x}$ where $\forall n, \quad \bar{x}(n) = T$ or F accordingly as $x(n) = F$ or T.

   The desired machine is shown in figure 2. In the state Y the output is the same as the input, in state N the output is the opposite of the input. From state $init$ the machine transitions non-deterministically to N or Y. After the first input, the machine stays in Y forever or in N forever. So for every input sequence there are two possible output sequences.

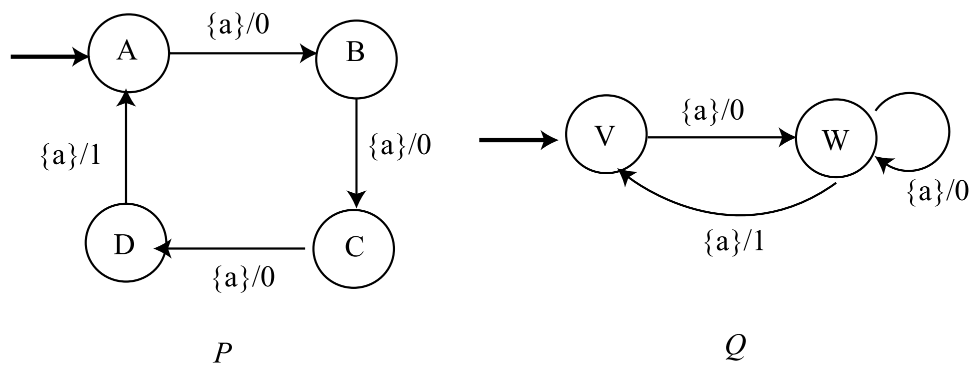

Figure 3: Q simulates P

5. **10 points** Let

$$M = (\textit{States}_M, \textit{Inputs}, \textit{Outputs}, \textit{possibleUpdates}_M, \textit{initialState}_M),$$

$$N = (\textit{States}_N, \textit{Inputs}, \textit{Outputs}, \textit{possibleUpdates}_N, \textit{initialState}_N),$$

be two non-deterministic state machines with the same set of inputs and outputs. Let $S \subset \textit{States}_M \times \textit{States}_N$.

(a) **4 pts** Give the definition for S to be a simulation relation.
S is a simulation relation if:

- $(\textit{initialState}_M, \textit{initialState}_N) \in S$;
- $\forall (s_M, s_N) \in S, \forall x \in \textit{Inputs}$, whenever

$$(s'_M, y) \in \textit{possibleUpdates}_M(s_M, x)$$

$\exists (s'_N, y) \in \textit{possibleUpdates}_N(s_N, x)$ such that $(s'_M, s'_N) \in S$.

(b) **4 pts** Find the simulation relation between P and Q shown in figure 3. Here $\textit{Inputs} = \{a\}$ and $\textit{Outputs} = \{0, 1\}$. (In the figure M is deterministic.)
The simulation relation is

$$S = \{(A, V), (B, W), (C, W), (D, W)\}.$$

(c) **2 pts** Are P and Q in figure 3 bisimilar? Answer yes or no.
No.

6. **10 points** Consider a multidimensional SISO system

$$\begin{aligned} s(n+1) &= As(n) + bx(n) \\ y(n) &= c^T s(n) + dx(n) \end{aligned}$$

Suppose you don't know A, b, c, d or the initial state $s(0)$. Two input-output experiments are performed. In the first experiment, the input signal is x and the output signal is y; in the second, the input signal is v and the output signal is w. These signals are shown in figure4. In both cases the initial state $s(0)$ is the same.

(a) **4 pts** What is the zero-state impulse response of the system?

We know that $\forall n \in$ *Ints*,

$$y(n) = c^T A^n s(0) + (h * x)(n) \quad (1)$$

$$w(n) = c^T A^n s(0) + (h * v)(n) \quad (2)$$

Subtracting (2) from (1) gives

$$(y - w)(n) = h * (x - v)(n).$$

But from the figure, $x - v = \delta$, the Kronecker delta, so $h * (x - v) = h * \delta = h$. So the zero-state impulse response is

$$h(n) = (y - w)(n) = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases} \quad (3)$$

(b) **3 pts** What is the zero-state step response, i.e. the zero-state response of the system to the input signal x?

The zero-state step response is

$$(h * x)(n) = \sum_{i=0}^{n} h(i)x(n-i) = \sum_{i=0}^{n} 1 = \begin{cases} 0, & n < 0 \\ n+1, & n \geq 0 \end{cases} \quad (4)$$

(c) **3 pts** What is the zero-input response, i.e. the response when the input signal is identically zero ($s(0)$ is still the initial state).

The zero-input response is $c^T A^n s(0)$. From (1) and (4) we get for $n \geq 0$,

$$c^T A^n s(0) = y(n) - (h * x)(n) \equiv 0.$$

(So in this case, $s(0)$ may be 0.)

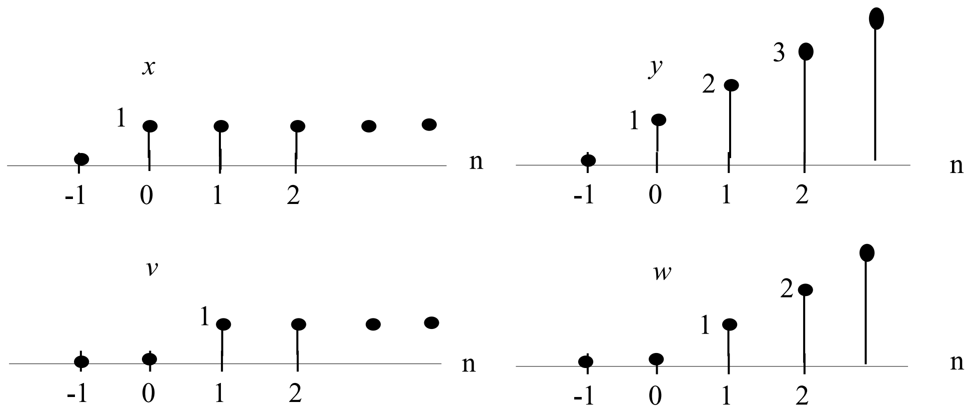

Figure 4: Results of two experiments

7. **10 points** Answer the following True/False questions about a system

$$H : [\mathit{Ints} \to \mathit{Reals}] \to [\mathit{Ints} \to \mathit{Reals}]$$

In each case a correct answer yields +2 points, an incorrect answer yields -2 points, no answer yields 0 points.

(a) If

$$\forall x, \forall n, \quad (H(x))(n) = x(-n), \tag{5}$$

H is linear. – TRUE

(b) The system (5) is time-invariant. – FALSE

(c) If

$$\forall x, \forall n, \quad (H(x))(n) = x^2(n) - x^2(n-1), \tag{6}$$

H is linear. – FALSE

(d) The system (6) is time-invariant. – TRUE

(e) The system given by

$$\forall x, \forall n, \quad (H(x))(n) = 0.5x(n) + 0.2x(n-3),$$

is linear and time-invariant. – TRUE

8. **10 points** Construct a linear time-invariant system of the form,

$$\begin{aligned} s(n+1) &= As(n) + bx(n) \\ y(n) &= c^T s(n) + dx(n), \end{aligned}$$

whose zero-state impulse response h is given by: $h(0) = 3, h(1) = -2$, and $h(n) = 0$, otherwise.

The zero-state response is $y(n) = h(1)x(n-1) + h(0)x(n)$, so the state only needs to remember the previous input, i.e. we only need a 1-dimensional state $s(n) = x(n-1)$. The required system is:

$$\begin{aligned} s(n+1) &= 0s(n) + 1.x(n) \\ y(n) &= -2s(n) + 3.x(n) \end{aligned}$$

So $A = 0, b = 1, c = -2, d = 3$.