EECS 20. Solutions to Midterm No. 2. November 12, 1999.

1. **20 points** Let $x : Reals \to Comps$ be a continuous-time signal with Fourier Transform X. The **bandwidth** of x is defined to be the smallest number Ω_x (rads/sec) such that $|X(\omega)| = 0$ for $|\omega| > \Omega_x$. If there is no such finite number, $\Omega_x = \infty$.

   Answer the following and give a brief justification for your answer.

   (a) If $\forall t, x(t) = 1$, what is X and what is the bandwidth of x?
   $\forall \omega, X(\omega) = 2\pi\delta(\omega)$, and so $\Omega_x = 0$.

   (b) If $\forall t, x(t) = \delta(t)$ (Dirac delta), what is X and what is the bandwidth of x?
   $\forall \omega, X(\omega) = 1$, and so $\Omega_x = \infty$.

   (c) If $\forall t, x(t) = \cos(t)$, what is X and what is the bandwidth of x?
   $\forall \omega, X(\omega) = \pi[\delta(\omega - 1) + \delta(\omega + 1)]$, and so $\Omega_x = 1$.

   (d) If x has bandwidth Ω_x what is the bandwidth of the signal $2x$?
   Since the Fourier Transform of $2x$ is $2X$, $\Omega_{2x} = \Omega_x$.

   (e) If x has bandwidth Ω_x and y has bandwidth $\Omega_y > \Omega_x$, what is the bandwidth of the signal $x + y$?
   Since the Fourier Transform of $x + y$ is $X + Y$, $\Omega_{x+y} = \Omega_y$.

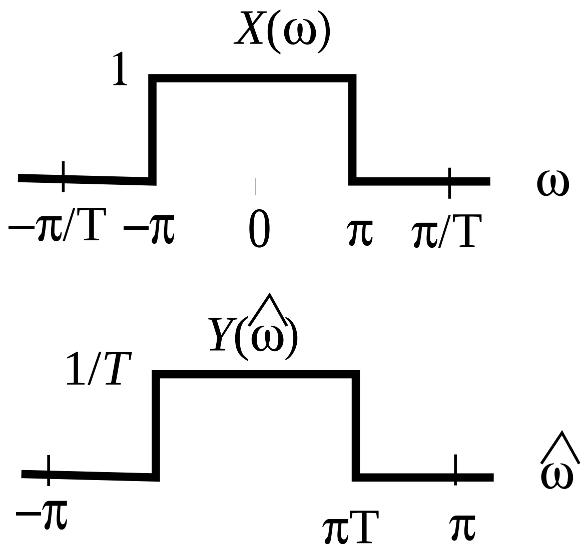

Figure 1: X Y for problem 2

2. **30 points** Suppose x is a continuous-time signal, with Fourier Transform X.

   (a) What are the units of ω in $X(\omega)$?
   radians/second.

   (b) Write down the definition of $y = Sampler_T(x)$.
   $y : Ints \to Comps$ and $\forall n \in Ints, y(n) = x(nT)$.

   (c) Let $Y(\hat{\omega})$ be the DTFT of y. What are the units of $\hat{\omega}$?
   radians/sample.

   (d) What is Y in terms of X?

   $$Y(\hat{\omega}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X(\frac{\hat{\omega} - 2\pi k}{T}).$$

   (e) Suppose X is as shown in Figure 1. For what values of T will there be no aliasing?
   The bandwidth of x is π, so there is no aliasing if and only if $\pi < \pi/T$ or $T < 1$.

   (f) Sketch $Y(\hat{\omega})$ when $T = 1/2$ and when $T = 3/4$?
   Since in both cases $T < 1$, there is no aliasing, and we can use the same sketch for Y as shown in the lower part of Figure 1. Note: Y is periodic with period 2π and the figure shows only one period.

3. **30 points** Let $x : Ints \to Reals$ be a discrete-time signal with DTFT X. Let $h : Ints \to Reals$ be another signal with DTFT H. Let $y = h * x$, the convolution sum of h and x.

(a) Give an expression for y in terms of h and x.

$$\forall n, \quad y(n) = \sum_{m=-\infty}^{\infty} h(m)x(n-m).$$

(b) Let X, H, Y be the DTFT of x, h, and y, respectively. Express Y in terms of X, H.

$$\forall \omega, \quad Y(\omega) = H(\omega)X(\omega).$$

(c) Suppose

$$X(\omega) = \begin{cases} 1, & |\omega| \leq \pi/4 \\ 0, & \pi/4 < |\omega| \leq \pi \end{cases}$$

Find the signal x.

$x = InverseDTFT(X)$, i.e.

$$\begin{aligned} \forall n, \quad x(n) &= \tfrac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{i\omega n}d\omega = \tfrac{1}{2\pi}\int_{-\pi/4}^{\pi/4} e^{i\omega n}d\omega = \tfrac{1}{2\pi}\tfrac{e^{i\omega n}}{in}\Big|_{-\pi/4}^{\pi/4} \\ &= \tfrac{\sin \pi n/4}{\pi n} \end{aligned}$$

(d) Suppose

$$H(\omega) = \begin{cases} 0, & |\omega| \leq \pi/4 \\ 1, & \pi/4 < |\omega| \leq \pi \end{cases}$$

Find $y = h * x$?

Since for all ω, $Y(\omega) = H(\omega)X(\omega) = 0$, therefore

$$\forall n, \quad y(n) = 0.$$

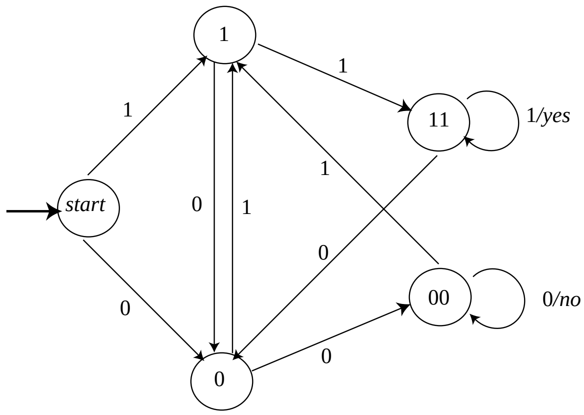

Figure 2: State machine for Problem 4

4. **20 points** Construct a state machine with $Inputs = \{0, 1\}$, $Outputs = \{Yes, No, absent\}$, such that for any input signal, the machine outputs Yes if the most recent three input values are 111, outputs No if the most recent three input values are 000, and in all other cases it outputs $absent$. In other words, if the input signal is

$$u(0), u(1), \cdots,$$

then the output signal

$$y(0), y(1), \cdots,$$

where

$$y(n) = \begin{cases} yes, & \text{if } (u(n-2), u(n-1), u(n)) = 111 \\ no, & \text{if } (u(n-2), u(n-1), u(n)) = 000 \\ absent, & \text{otherwise} \end{cases}$$

The state machine needs to remember four patterns: 0,00,1,11. It is given by the diagram in Figure 2. Note that if no output is explicitly declared, it is $absent$.