EECS 20. Final Exam Solutions 15 May 1999

1. **15 points** Answer these short questions and use the space below for your calculations.

   (a) The solutions of the equation $e^{j4\theta} = 1$ are $\theta =$
   Ans $\theta = 0, \pi/2, \pi, 3\pi/2$.

   (b) Express $\cos 3\theta$ and $\sin 3\theta$ in terms of $\cos\theta$ and $\sin\theta$:

   $$\cos 3\theta =$$

   $$\sin 3\theta =$$

   Ans

   $$\begin{aligned} \cos 3\theta + j\sin 3\theta &= e^{j3\theta} = [\cos\theta + j\sin\theta]^3 \\ &= [\cos^3\theta - 3\cos\theta\sin^2\theta] + j[3\cos^2\theta\sin\theta - \sin^3\theta \end{aligned}$$

   So,

   $$\begin{aligned} \cos 3\theta &= \cos^3\theta - 3\cos\theta\sin^2\theta \\ \sin 3\theta &= [3\cos^2\theta\sin\theta - \sin^3\theta] \end{aligned}$$

   (c) For what *real-valued* numbers ω is the function x periodic:

   $$\forall n \in Ints,\ x(n) = \cos\omega n$$

   and what is the period?
   Ans x is periodic with integer period p provided that $\omega(n+p) = \omega n + 2\pi m$, or $\omega p = 2\pi m$, or $\omega = 2\pi m/p$ for some integer m.

   (d) The general form of the following matrix for $n \geq 0$ is:

   $$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^n =$$

   Ans

   $$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$$

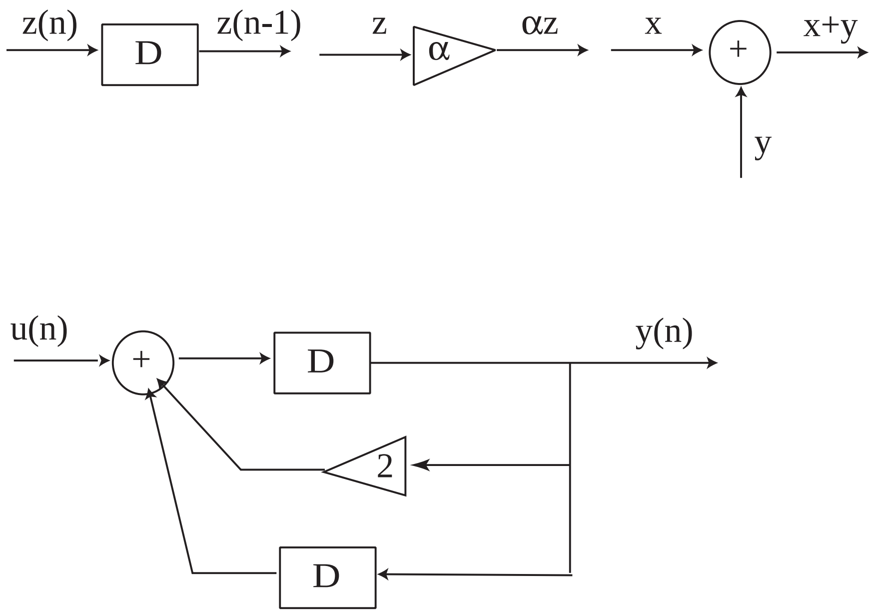

Figure 1: An LTI system can be built using unit delays, gains, and adders

2. **15 points** A LTI system can be built using unit delay elements D, gains α, and adders, shown on top of Figure 1.

   (a) Express the relation between the input and output of the system in the lower part of the figure in the form:

   $$y(n) = a_1 y(n-1) + \cdots + a_k y(n-k) + b_1 u(n-1) + \cdots b_m u(n-m),$$

   i.e. determine k, m and the coefficients a_i, b_j for the system in the figure.

   (b) Determine the frequency response $H(\omega)$ of this system using the fact that $y = H(\omega)u$ when u is given by $\forall n,\ u(n) = e^{j\omega n}$.

Ans From Figure 2 we can see that

$$\forall n,\ y(n) = 2y(n-1) + y(n-2) + u(n-1) \tag{1}$$

So $k = 2$, $m = 1$, $a_1 = 2, a_2 = 1, b_1 = 1$.

Suppose $\forall n,\ u(n) = e^{j\omega n}, y(n) = H(\omega)e^{j\omega n}$. Substituting in (1) gives

$$H(\omega)e^{j\omega n} = 2H(\omega)e^{-j\omega}e^{j\omega n} + H(\omega)e^{-2j\omega}e^{j\omega n} + e^{-j\omega}e^{j\omega n}$$

So

$$H(\omega) = \frac{e^{-j\omega}}{1 - 2e^{-j\omega} - e^{-2j\omega}}$$

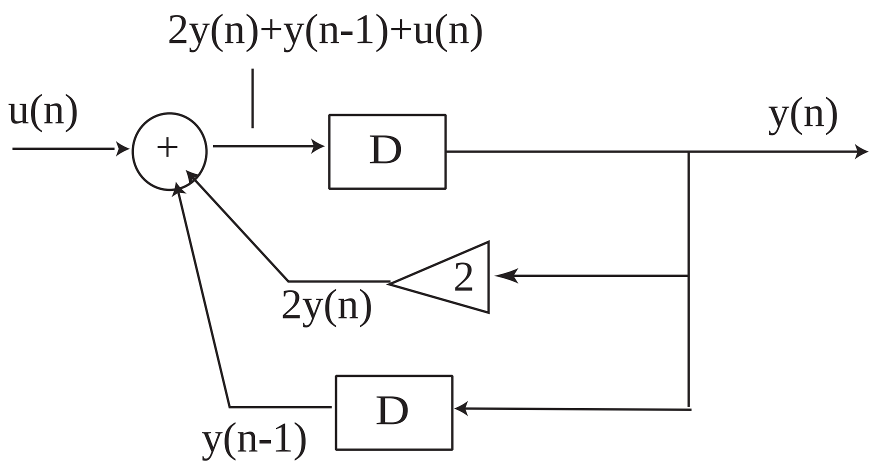

Figure 2: System of Figure 1

3. **15 points** Consider the difference equation system:

$$\forall n,\ y(n) = 0.5y(n-1) + u(n-1). \tag{2}$$

   (a) What is the zero-state impulse response of this system?

   (b) Use this result to obtain the zero-state impulse response of the system:

$$\forall n,\ y(n) = 0.5y(n-1) + u(n-1) + u(n-2). \tag{3}$$

Ans The zero-state impulse response is

$$h(n) = \begin{cases} (0.5)^{n-1}, & n \geq 1 \\ 0, & n < 1 \end{cases}$$

The zero-state impulse response of (3) is the same as the zero-state response of (2) to the input u given by

$$\forall n,\ u(n) = \delta(n) + \delta(n-1)$$

and since the system is LTI, the response is

$$\forall n,\ h(n) + h(n-1)$$

where h is given above.

4. **15 points** Consider the moving average system (with input x and output y)

$$\forall t \in \textit{Reals}, \; y(t) = \int_{-0.5}^{0.5} x(t-s)ds.$$

(a) What is the impulse response h of this system?

(b) What is its frequency response?

(c) Use the previous result to determine the response y when the input is $\forall t, \; x(t) = \sin(\omega t)$.

Ans The impulse response h is the response to the Dirac delta function, so, using the sifting property,

$$\begin{aligned} \forall t, \; h(t) &= \int_{s=-0.5}^{0.5} \delta(t-s)ds \\ &= \begin{cases} 1, \text{ if } -0.5 < t < 0.5 \\ 0, \text{ otherwise} \end{cases} \end{aligned}$$

The frequency response $H = FT(h)$,

$$\begin{aligned} H(\omega) &= \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt = \int_{-0.5}^{0.5} h(t)e^{-j\omega t}dt \\ &= \frac{e^{-j\omega t}}{-j\omega}|_{t=-0.5}^{t=0.5} = \frac{\sin 0.5\omega}{0.5\omega} \end{aligned}$$

To find the frequency response to the signal $\forall t, \; x(t) = \sin \omega t$ we write x as

$$x(t) = \frac{1}{2j}[e^{j\omega t} - e^{-j\omega t}],$$

so the response is

$$\begin{aligned} y(t) &= \frac{1}{2j}[H(\omega)e^{j\omega t} - H(-\omega)e^{-j\omega t}] \\ &= \frac{1}{2j}[\frac{\sin 0.5\omega}{0.5\omega}e^{j\omega t} - \frac{\sin(-0.5\omega)}{-0.5\omega}e^{-j\omega t} \\ &= \frac{\sin 0.5\omega}{0.5\omega}\sin \omega t. \end{aligned}$$

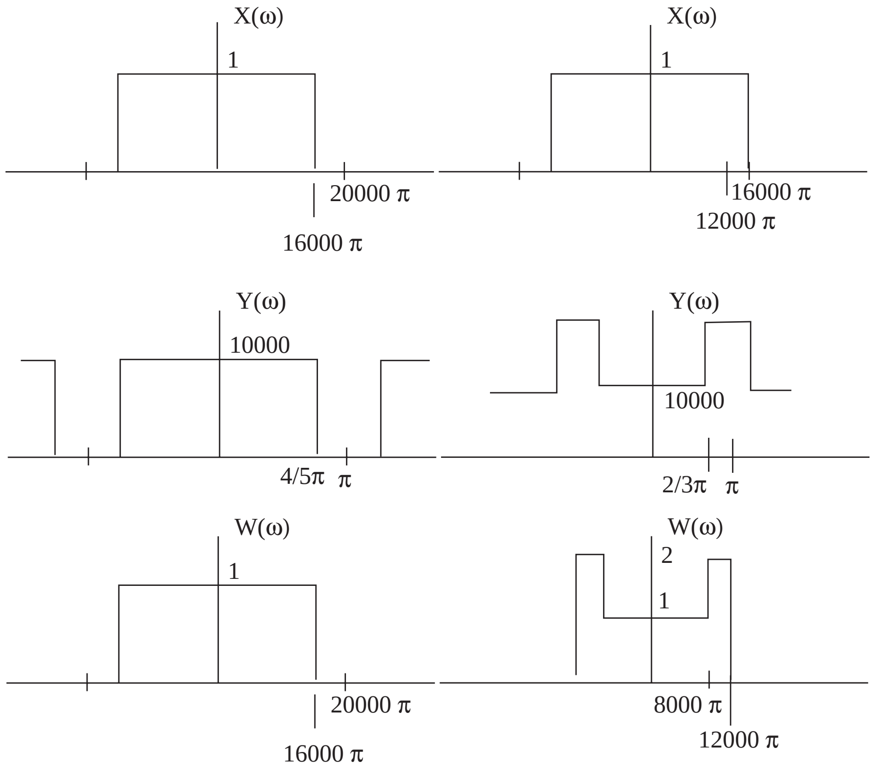

Figure 3: The graphs on the left are for $T = 1/20000$; the graphs on the right are for $T = 1/12000$.

5. **15 points** Let x be a continuous-time signal with Fourier Transform $X = FT(x)$, with

$$X(\omega) = \begin{cases} 1, & |\omega| < 2\pi \times 8,000 \text{ rads/sec} \\ 0, & \text{otherwise} \end{cases}$$

Let $y = \mathit{Sampler}_T(x)$, $Y = DTFT(y)$. Let $w = \mathit{IdealInterpolator}_T \circ \mathit{Sampler}_T(x)$, and $W = FT(w)$.

(a) Sketch X, Y, and W for $T = 1/20,000$ sec and $T = 1/12,000$ sec.

(b) For what values of T is $x = w$?

Ans From Chapter 9, we know that Y is periodic with period 2π,

$$Y(\omega) = \frac{1}{T}\sum_{-\infty}^{\infty} X(\frac{\omega - 2\pi k}{T}), \ |\omega| < \pi \quad (4)$$

$$W(\frac{\omega}{T}) = \begin{cases} TY(\omega), & |\omega/T| < \pi \\ 0, & |\omega/T| > \pi \end{cases} \quad (5)$$

For $T = 1/20000$ there is no aliasing, and $W = X$. For $T = 1/12000$ there is aliasing, and so Y, W are as shown.

There is no aliasing, $W = X$, if and only if $T < 1/16000$.

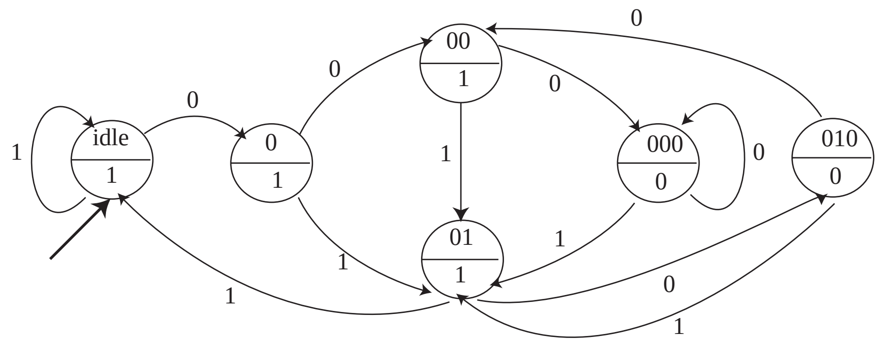

Figure 4: The machine that realizes H

6. Construct a state machine with $U = Y = \{0, 1\}$ whose response function is: If $H(u) = y$, then

$$\forall n \geq 0,\ y(n) = \begin{cases} 0, & \text{if } u(n-3), u(n-2), u(n-1) = 000 \text{ or } 010 \\ 1, & \text{otherwise} \end{cases}$$

Ans The machine of Figure 4 does the job.

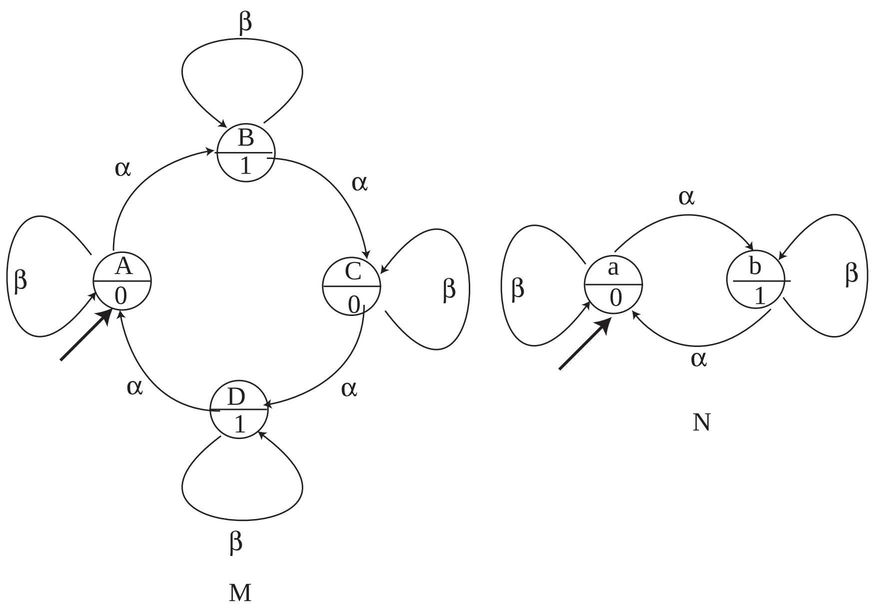

Figure 5: The machine N simulates machine M

7. **15 points** Find a simulation relation S and show that N simulates M.

   Ans The simulation relation is: $S = \{(A, a), (C, a), (B, b), (D, b)\}$. We can see from the figure that if $(x_1, x_2) \in S$, then the output in x_1 (in M) is the same as the output in x_2 (in M_2). And if $f_1(x_1, u) = x_1'$ and $f_2(x_2, u) = x_2'$, then $(x_1', x_2') \in S$. Finally, the initial states satisfy $(A, a) \in S$. So N simulates M.