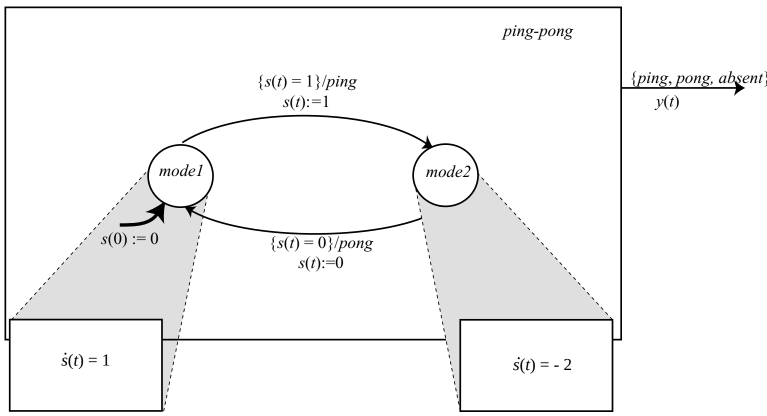
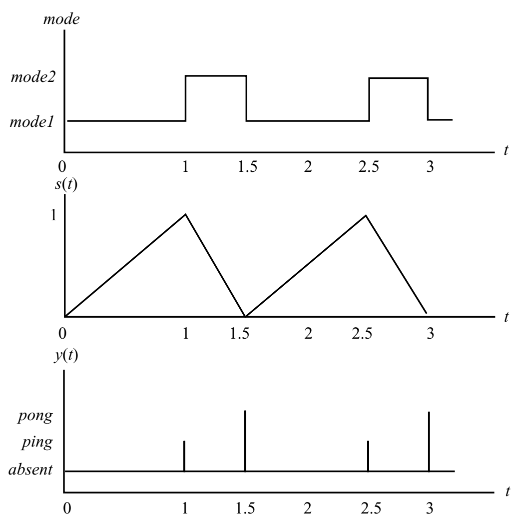
**EECS 20. Midterm No. 2 Solution**
**April 11, 2003.**

1. **15 points** For the following hybrid system sketch

   (a) **10 points** the state trajectory (both the mode and the continuous state) and

   (b) **5 points** the output signal for $0 \le t \le 3$.




2. **15 points, 5 points each part**
   **Give the units of period and frequency below**

   (a) Consider the discrete-time signal $x$ given by

   $$\forall n \in Integers, \quad x(n) = \cos \omega n.$$

   For what values of $\omega$ is $x$ periodic, and what is the period?
   $x$ is periodic with period $\boxed{p \text{ samples}}$ if $\omega p$ is a multiple of $2\pi$, i.e. if $\boxed{\omega = 2m\pi/p \text{ rad/sample}}$ for integers $m, p$. To get the smallest period $p$, $m, p$ must be coprime.

(b) Consider the discrete-time signal $x$ given by

$$\forall n \in Integers, \quad x(n) = 1 + \cos(4\pi n/9).$$

What is its period $p$ and what is its fundamental frequency?

The period is $\boxed{p = 9 \text{ samples}}$, and the fundamental frequency is $\boxed{\omega_0 = 2\pi/9 \text{ rads/sample}}$.

The signal has the Fourier series representation

$$\forall n, \quad y(n) = A_0 + \sum_{k=1}^{\lfloor p/2 \rfloor} A_k \cos(k\omega_0 n + \phi_k).$$

Identify $\omega_0, A_0, A_k, \phi_k$.

$\boxed{A_0 = 1, A_2 = 1, \phi_2 = 0, \text{ other coefficients are 0}}$.

(c) Consider the continuous-time periodic signal $y$ given by

$$\forall t \in Reals, \quad y(t) = \cos 5t + \sin 3t.$$

What is its period and what is its fundamental frequency?

Its fundamental frequency is $\boxed{\omega_0 = gcd\{3, 5\} = 1 \text{ rad/sec}}$ and the period is $\boxed{p = 2\pi/p = 2\pi \text{ sec}}$

The Fourier series representation of $y$ is

$$\forall t, \quad y(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \phi_k).$$

Identify $\omega_0$, $A_0$, $A_k, \phi_k$.

We have

$$\forall t, \quad y(t) = \cos 5t + \cos(3t - \pi/2),$$

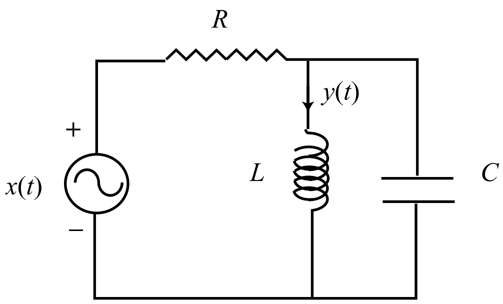
So,

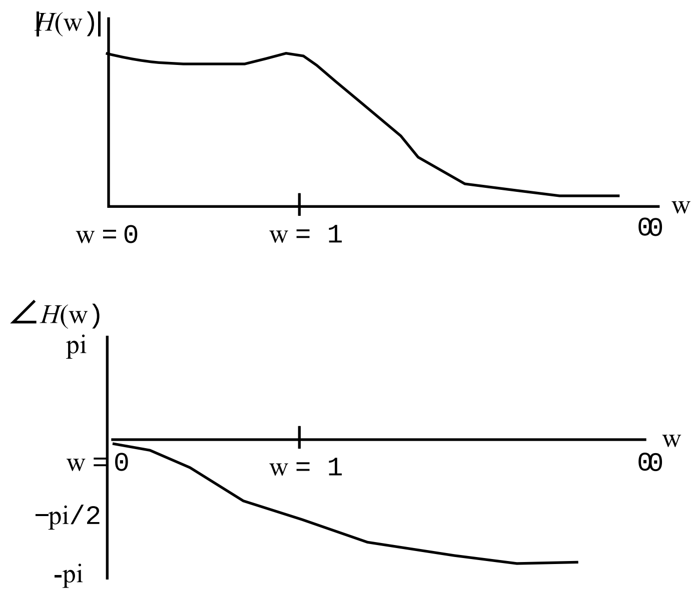
$$\boxed{A_3 = 1, \phi_3 = -\pi/2, A_5 = 1, \phi_5 = 0, \text{ other coefficients are 0}}$$

3. **15 points, 3 points each part** Consider the following discrete-time systems with input $x : Integers \to Reals$ and output $y : Integers \to Reals$. For each system, state whether it is linear (L), time-invariant (T), both (LTI), or neither (N).

   (a) $\forall n, \quad y(n) = x(-n)$. **L**

   (b) $\forall n, \quad y(n) = [x(n) + x(n-1)]^2$. **TI**




(c) $\forall n, \quad y(n) = n[x(n) + x(n-1)]$. **L**

(d) $\forall n, \quad y(n) = x(2n)$. **L**

(e) $\forall n, \quad y(n) = [x(n) + x(n+1)]/2$. **LTI**

4. **20 points** The $R, L, C$ circuit in the figure has for its input signal the voltage $x$ and its output signal is the inductor current $y$. From Kirchhoff's law one can determine that these signals are related by the differential equation

$$\forall t, \quad RLC\frac{d^2y(t)}{dt^2} + L\frac{dy(t)}{dt} + Ry(t) = x(t).$$

(a) **6 points** Find the frequency response $H : Reals \to Complex$ of this system.
The frequency response is obtained by setting $\forall t, x(t) = e^{i\omega t}, y(t) = H(\omega)e^{i\omega t}$, substituting in the differential equation, to get

$$\forall \omega \in Reals, \quad H(\omega) = \frac{1}{R - RLC\omega^2 + iL\omega}.$$

(b) **7 points** Obtain an expression for the amplitude response and the phase response, assuming $R = L = C = 1$.
Substituting gives

$$H(\omega) = \frac{1}{(1-\omega^2) + i\omega},$$

which in polar coordinates gives the amplitude response

$$|H(\omega)| = \frac{1}{[(1-\omega^2)^2 + \omega^2]^{1/2}},$$

and the phase response

$$\angle H(\omega) = -\tan^{-1}\frac{\omega}{1-\omega^2}.$$

(c) **7 points** Sketch the amplitude response and the phase response. Carefully mark the values for $\omega = 0, 1$ and $\omega \to \infty$.

We have

$$H(0) = 1,\ H(1) = \frac{1}{i} = e^{-i\pi/2},\ \lim_{\omega\to\infty}|H(\omega)| = 0,\ \lim_{\omega\to\infty}\angle H(\omega) = -\pi.$$

5. **15 points, 5 points each part** Fill in the blanks:

(a) The five roots of $z^5 = 1$ are:

$$\boxed{z = e^{2n\pi/5}, n = 0, 1, 2, 3, 4.}$$

(b) $\forall t, \cos(\omega t) + \cos(\omega t + \pi/2) = ReAe^{i[\omega t + \phi]}$

in which $\boxed{A = \sqrt{2}, \phi = \pi/4.}$

(c) The polar representation of the following numbers are:

$1 + i = \sqrt{2}e^{i\pi/4}$

$1 - i = \sqrt{2}e^{-i\pi/4}$

$[1 + i]^{-1} = \frac{1}{\sqrt{2}}e^{-i\pi/4}$