EECS 20. Solutions to Final Exam. December 16, 1999.

1. **20 points** A signal is mathematically described as a function. As we have seen in the course, these may be functions of time and space or they may be sequences. For example, a one-second long soundwave may be described as a function $\mathit{Sound} : [0, 1] \to \mathit{AirPressure}$. Propose mathematical descriptionss for the signals corresponding to the following intuitive descriptions. Give a brief justification for your answer.

   (a) **5** The signal obtained after sampling *Sound* (described above) 8,000 times.

   (b) **5** A black-and-white 800×600 pixel image.

   (c) **5** A one-second long black-and-white video with 30 frames per second.

   (d) **5** The sequence of buttons you press with your TV remote control.

   Answer
   (a) We have

   $$\mathit{SampledSound} : \{0, 1, \cdots, 8,000\} \to \mathit{AirPressure},$$

   where for all n, $\mathit{SampledSound}(n) = \mathit{Sound}(n/8,000)$.
   (b) We have

   $$\mathit{Image} : \{0, \cdots, 799\} \times \{0, \cdots, 599\} \to \{0, \cdots, 255\},$$

   where the range is an 8-bit color-map index.
   (c) We have

   $$\mathit{Video} : \{0, \cdots, 29\} \to \mathit{Images},$$

   where *Images* are all functions of the form *Image*.
   (d) The sequence could be modeled as

   $$\mathit{Ints} \to \{1, \cdots, 5, \mathit{open}, \mathit{close}\},$$

   assuming that the available buttons are: $1, \cdots, 5, \mathit{open}, \mathit{close}$.

2. **20 points**

   (a) **5** Write $\cos^3(\omega t)$ in terms of cosines and sines of multiples of ωt.

   (b) **5** Express in polar form all the **distinct** roots of the equation $z^5 = 2$.

   (c) **5** Find A and θ so that

   $$A \sin(\omega t + \theta) = \sin(\omega t) + \cos(\omega t).$$

   (d) **5** Express the fraction $\frac{1+j\omega}{1-j\omega}$ in rectangular and polar forms.

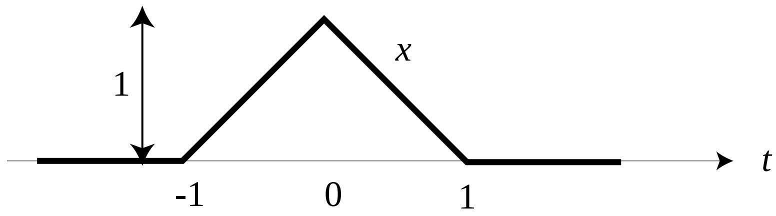

Figure 1: The graph of x for problem 3

Answer
(a) Write

$$\begin{aligned}\cos^3(\omega t) &= 1/8[e^{i\omega t} + e^{-i\omega t}]^3 \\ &= 1/8[e^{3i\omega t} + e^{-3i\omega t} + 3e^{i\omega t} + 3e^{-\omega t}] \\ &= 1/4[\cos(3\omega t) + 3\cos(\omega t).]\end{aligned}$$

(b) The distinct roots are:

$$\{2^{1/5} \times e^{i2\pi k/5} \mid k = 0, 1, 2, 3, 4\}.$$

(c) Since

$$\sin(\omega t + \theta) = \sin(\omega t)\cos(\theta) + \cos(\omega t)\sin(\theta) = \sin(\omega t) + \cos(\omega t),$$

we must have $A\cos(\theta) = A\sin(\theta) = 1$ which gives $A = \sqrt{2}, \theta = \pi/4$.
(d) We have

$$\frac{1+j\omega}{1-j\omega} = \frac{(1+j\omega)^2}{1+\omega^2} = \frac{1-\omega^2}{1+\omega^2} + j\frac{2\omega}{1+\omega^2},$$

and

$$\frac{1+j\omega}{1-j\omega} = 1 \times e^{j2\tan^{-1}\omega}.$$

3. **20 points** The function x : *Reals* $\to$ *Reals* is described by its graph in Figure 1. Define the periodic signal y : *Reals* $\to$ *Reals* by

$$\forall t, \quad y(t) = \sum_{k=-\infty}^{\infty} x(t - kp).$$

 (a) **3** What is the period of y?
 (b) **4, 4** Carefully sketch y for $p = 2$ and $p = 4$.

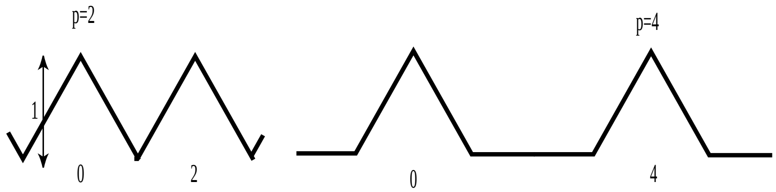

Figure 2: The graph of y for problem 3

(c) **4** Since y is periodic, it has an exponential Fourier series expression

$$\forall t, \quad y(t) = \sum_{k=-\infty}^{\infty} Y_k e^{jk\omega_0 t}. \tag{1}$$

What is ω_0? What are its units?

(d) **5** Compute the Fourier series coefficient Y_0 in (1) for the case $p = 2$ and $p = 4$.

Answer
(a) The period is p seconds.
(b) The sketches are shown in Figure 2.
(c) $\omega_0 = 2\pi/p$ radians/second.
(d) We have $Y_0 = 1/p \int_0^p y(t)dt$. So $Y_0 = 1/2$ if $p = 2$, and $Y_0 = 1/4$ if $p = 4$.

4. **20 points** Consider an LTI system L with impulse response h given by

$$\forall\ n \in \mathit{Ints}, \quad h(n) = \delta(n+2) + \delta(n) + \delta(n-2),$$

where δ is the Kronecker delta function.

(a) **10** Find the frequency response H as a function of ω, frequency in radians per sample.

(b) **10** Find **all** values of $\omega \in \mathit{Reals}$ such that if the input is the signal x where

$$\forall\ n \in \mathit{Ints}, \quad x(n) = \cos(\omega n),$$

then the output y will be zero, i.e.

$$\forall\ n \in \mathit{Ints}, \quad y(n) = 0.$$

Answer
(a) We know that H is the DTFT of h:

$$H(\omega) = \sum_n h(n) e^{-i\omega n} = 1 + e^{i2\omega} + e^{-i2\omega} = 1 + 2\cos(2\omega).$$

(b) We know that since $x(n) = \cos(\omega n)$, $y(n) = H(\omega)\cos(\omega n) = 0$ if and only if $H(\omega) = [1 + 2\cos(2\omega)] = 0$. So ω must satisfy $\cos(2\omega) = -1/2$.

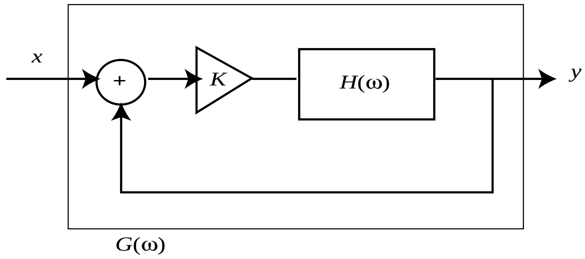

Figure 3: Feedback arrangement for Problem 5

5. **20 points** A continuous-time LTI system with frequency response $H(\omega)$ is put in the feedback arrangement of Figure 3. Here K is a constant gain.

   (a) **10** Determine the frequency response $G(\omega)$ of the feedback system in terms of H and K.

   (b) **10** Suppose $K = 1$ and $H(\omega) = 1/(1 + j\omega)$. Find G.

   Answer
   (a) We have

$$G(\omega) = \frac{KH(\omega)}{1 - KH(\omega)}.$$

   (b) Substituting for K, H gives

$$G(\omega) = \frac{1/(1 + j\omega)}{1 - 1/(1 + j\omega)} = \frac{1}{j\omega}.$$

6. **30 points** The input signal u and output signal y of a continuous-time LTI system H are related by

$$\forall t, \quad \dot{y}(t) + y(t) = u(t).$$

   (a) **10** Find its frequency response $\hat{H}$. Plot the amplitude and phase response of $\hat{H}$.

   (b) **10** Find the Fourier Transform of the function

$$\forall t, \quad x(t) = \begin{cases} e^{-t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

   (c) **10** Use the result of the previous part to find the impulse response h of the system H.

   Answer
   (a) Taking $u(t) = e^{i\omega t}$ and $y(t) = \hat{H}(\omega)e^{i\omega t}$ gives

$$\hat{H}(\omega)i\omega e^{i\omega t} + \hat{H}(\omega)e^{i\omega t} = e^{i\omega t}.$$

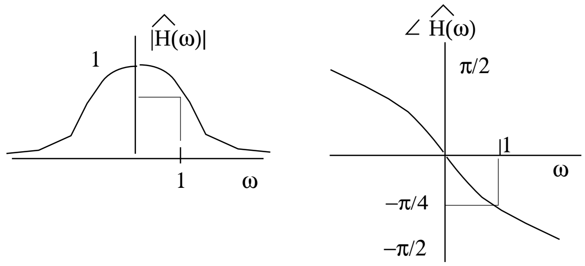

Figure 4: Magnitude and phase plots for problem 6

And so,

$$\hat{H}(\omega) = \frac{1}{1+i\omega} = \frac{1}{(1+\omega^2)^{1/2}} e^{-i\tan^{-1}(\omega)}.$$

So $|\hat{H}(\omega) = 1/[1+\omega^2]^{1/2}$ and $\angle\hat{H}(\omega) = -\tan^{-1}(\omega)$. These plots are shown in Figure 4.

(b) The Fourier transform is

$$\begin{aligned} X(\omega) &= \int_0^\infty e^{-t-i\omega t} dt \\ &= \frac{1}{1+i\omega} e^{-(1+i\omega)t}|_0^\infty = \frac{1}{1+i\omega}. \end{aligned}$$

(c) From part (b) we note that $\hat{H}(\omega) = X(\omega)$ and since h is the inverse FT of $\hat{H}$ we conclude that

$$\forall t, \quad h(t) = x(t) = \begin{cases} e^{-t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

7. **30 points** Let $x : Reals \to Comps$ be a continuous-time signal with Fourier Transform X. The **bandwidth** of x is defined to be the smallest number Ω_x (rads/sec) such that $|X(\omega)| = 0$ for $|\omega| > \Omega_x$. If there is no such finite number, we say that the signal is not bandlimited and let $\Omega_x = \infty$.

   Answer the following and give a brief justification for your answer.

   (a) **4** If x has bandwidth Ω_x what is the bandwidth of the signal $2x$?

   (b) **4** If x has bandwidth Ω_x and y has bandwidth $\Omega_y > \Omega_x$, what is the bandwidth of the signal $x + y$?

   (c) **7** If x has bandwidth Ω_x what is the bandwidth of $x * x$, the convolution of x with itself?

   (d) **7** If x has bandwidth Ω_x what is the bandwidth of x^2?

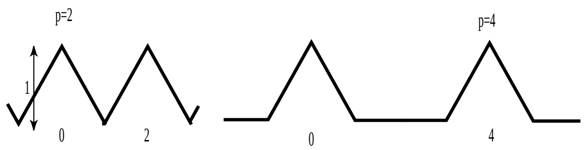

Figure 5: X for problem 8

(e) **4** If $\forall t, y(t) = x(t-1)$, what is the bandwidth of y?

(f) **4** If $\forall t, y(t) = x(2t)$, what is the bandwidth of y?

Answer
(a) $\Omega_{2x} = \Omega_x$, since FT of $2x$ is $2X(\omega)$.
(b) $\Omega_{x+y} = \Omega_y$, since FT of $x + y$ is $X(\omega) + Y(\omega)$.
(c) $\Omega_{x*x} = \Omega_x$, since FT of $x * x$ is $X^2(\omega)$.
(d) $\Omega_{x^2} = 2\Omega_x$, since the FT of x^2 is $(1/2\pi)X * X$.
(e) $\Omega_y = \Omega_x$, since $|Y(\omega)| = |X(\omega)|$.
(f) $\Omega_y = 2\Omega_x$, since $|Y(\omega)| = 1/2|X(\omega/2)|$.

8. **30 points** Suppose x is a continuous-time signal, with Fourier Transform X.

(a) **3** What are the units of ω in $X(\omega)$?

(b) **3** Write down the definition of $y = Sampler_T(x)$.

(c) **3** Let $Y(\hat{\omega})$ be the DTFT of y. What are the units of $\hat{\omega}$?

(d) **4** What is Y in terms of X?

(e) **7** Suppose X is as shown in Figure 5. For what values of T will there be no aliasing?

(f) **10** Sketch $Y(\hat{\omega})$ when $T = 1/2$ and when $T = 3/4$?

Answer
(a) ω is in radians/sec.
(b) For all $n \in Ints$,

$$y(n) = x(nT).$$

(c) $\hat{\omega}$ is in radians/sample.
(d) We have

$$Y(\hat{\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(\frac{\hat{\omega} - 2\pi k}{T}).$$

(e) There will be no aliasing provided $\pi/T > \pi$ or $T < 1$.
(f) See Figure 6. Y is periodic with period 2π and the figure shows Y for $-\pi < \hat{\omega} < \pi$.

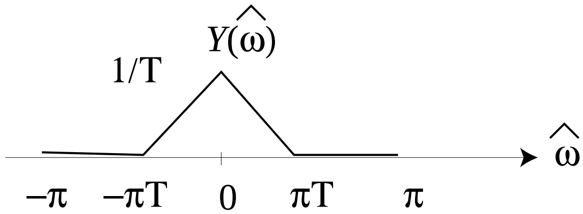

Figure 6: Y for problem 8

9. **20 points** A certain device is controlled via a keyboard that has only the alphabetic keys, A through Z. Assume that you can only press one key at a time (unlike a real computer keyboard). A portion of the specification this device says:

> To shut down the device, the user should hit keys "A", "B", and "C" in sequence, one at a time.

The designer of the device uses a state machine to get the desired behavior, and defines the state machine with the following table:

state	*next state under input*			
	A	B	C	D-Z
state1	state2			
state2		state3		state1
state3			stop	state1
stop				

The start state is *state1*. The blank entries indicate that the state machine stays in the same state. There is no output from this state machine. Notice that the response to the whole set of keys D through Z is given in a single column of the table, for compactness.

(a) **10** Sketch and label the state transition diagram corresponding to the above table. Be sure your diagram specifies all aspects of the behavior. In particular, show "else" transitions (if any) explicitly.

(b) **10** The table has a bug in it. Its behavior does not quite match the specification. Identify the bug; in particular, give a sequence of inputs that illustrates the bug, and give a state transition table that corrects the bug.

Answer
(a) The diagram is shown in Figure 7

(b) Under the input sequences $ABAC$ the sequence of states is 1,2,3,3,stop which is incorrect. The correct machine is given in the following table. The transitions from state 3 need to be changed.

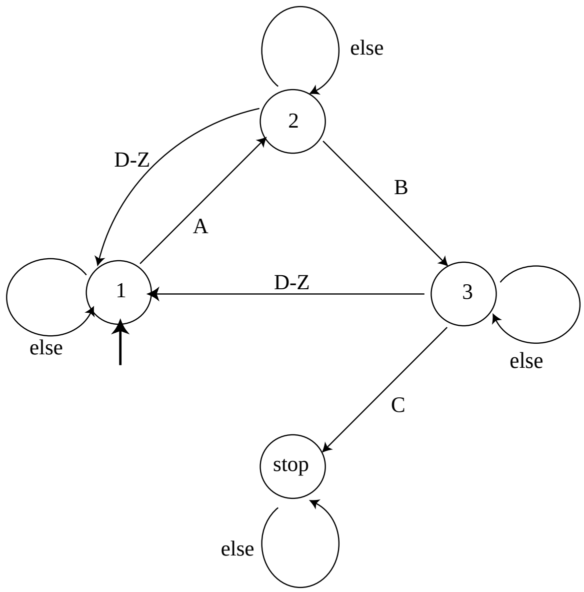

Figure 7: State machine for problem 9

state	*next state under input*			
	A	B	C	D-Z
state1	state2			
state2		state3	state1	state1
state3	state1	state1	stop	state1
stop				

10. **20 points** A single-input, single-output linear difference equation system is expressed in the form

$$\begin{aligned} x(n+1) &= Ax(n) + bu(n) \\ y(n) &= c'x(n) + du(n) \end{aligned}$$

where A is a $N \times N$ matrix, b, c are $N \times 1$ column vectors, $d, u(n), y(n)$ are scalars.

(a) **8** Write down the expression for the zero-state impulse response, i.e. what is the output when the initial state $x(0) = 0$, and $u(n) = \delta(n)$, the Kronecker delta function.

(b) **12** Suppose

$$A = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad c' = [0 \quad 1] \quad d = 1$$

Find the zero-state impulse response.
Hint: for all $n \geq 0$,

$$A^n = \begin{bmatrix} \cos(n\theta) & \sin(n\theta) \\ -\sin(n\theta) & \cos(n\theta) \end{bmatrix},$$

Answer
(a) The zero-state impulse response is:

$$\forall n, \quad h(n) = y(n) = \left\{ \begin{array}{ll} d, & n = 0 \\ c' A^{n-1} b, & n \geq 0 \end{array} \right.$$

(b) Substituting above and using the hint gives: $h(0) = 1$ and for $n \geq 1$

$$h(n) = [0 \quad 1] \left[\begin{array}{cc} \cos((n-1)\theta) & \sin((n-1)\theta) \\ -\sin((n-1)\theta) & \cos((n-1)\theta) \end{array} \right] \left[\begin{array}{c} 1 \\ 0 \end{array} \right] = -\sin((n-1)\theta).$$