EECS 20. Midterm 2. November 6, 1998
Solution

1) **24 points**. Consider a continuous-time signal x with the following finite Fourier series expansion: for all $t \in$ *Reals*,

$$x(t) = \sum_{k=0}^{4} \cos(k\omega_0 t)$$

where $\omega_0 = \pi/4$ radians/second. Define $Sampler_T$: *ContSignals* $\rightarrow$ *DiscSignals* to be a sampler with sampling interval T (in seconds). Define *IdealDiscToCont* : *DiscSignals* $\rightarrow$ *ContSignals* to be an ideal bandlimited interpolation system. I.e., given a discrete-time signal $y(n)$, it constructs the continuous-time signal w where for all $t \in$ *Reals*,

$$w(t) = \sum_{k=-\infty}^{\infty} y(nT)\, p(t - nT)$$

where the pulse p is the sinc function,

$$p(t) = \frac{\sin(\pi t / T)}{\pi t / T}$$

a) Give an upper bound on T (in seconds) such that x = *IdealDiscToCont* ($Sampler_T$ (x)).
b) Suppose that T = 4 seconds. Give a *simple* expression for $y = Sampler_T(x)$.
c) For the same T = 4 seconds, give a *simple* expression for w = *IdealDiscToCont* ($Sampler_T(x)$).

solution to problem 1:

a) The highest frequency term is the k = 4 term in the summation, which has frequency $4\omega_0 = \pi$ radians/second. By the Nyquist-Shannon sampling theorem, we have to sample at a sampling frequency at least twice this, or 2π radians/second or 1 Hz. This means that the sampling interval must be at most 1/(1 Hz) = 1 second.
b) With T = 4 seconds we are sampling too slowly to avoid aliasing distortion. Solution:

$$y(n) = \sum_{k=0}^{4} \cos(k\omega_0 nT) = \sum_{k=0}^{4} \cos(k\pi n) = 1 + (-1)^n + 1 + (-1)^n + 1 = \begin{cases} 5 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}.$$

c) The ideal reconstructed signal will be the simplest, smoothest signal that passes through the samples, which in this case is

$$w(t) = 3 + 2\cos(\pi t / 4).$$

2) **24 points**. Consider an LTI discrete-time system *Filter* with impulse response h where for all $n \in$ *Ints*,

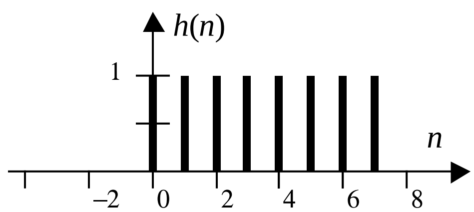
$$h(n) = \sum_{k=0}^{7} \delta(n-k)$$

where δ is the Kronecker delta function.

a) Sketch h.
b) Suppose the input signal x : *Ints* $\to$ *Reals* is such that for all $n \in$ *Ints*, $x(n) = \cos(\omega n)$, where $\omega = \pi/4$ radians/sample. Give a *simple* expression for y = *Filter*(x).
c) Give the value for $H(\omega)$ for $\omega = \pi/4$ radians/sample, where H = *DTFT*(h).

solution to problem 2:

a)



b) The convolution sum is

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k) = \sum_{k=0}^{7} x(n-k) = \sum_{k=0}^{7} \cos((n-k)\pi/4)$$

Notice that for any n this sum covers one complete cycle of the cosine, and thus adds to zero. Thus

$$y(n) = 0.$$

An alternative technique that is a bit more laborious is to find $H(\omega)$ and show that it is zero when $\omega = \pi/4$. To do this, write

$$H(\omega) = \sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k} = \sum_{k=0}^{7} e^{-j\omega k}$$

$$= e^{-j\pi 0/4} + e^{-j\pi 1/4} + e^{-j\pi 2/4} + e^{-j\pi 3/4} + e^{-j4/4} + e^{-j\pi 5/4} + e^{-j\pi 6/4} + e^{-j\pi 7/4}.$$

The easiest way to see that this is zero is to observe that this is the sum of a set of complex vectors arranged evenly in a circle. More formally, observe that

$$e^{-j\pi 5/4} = e^{-j\pi 4/4}e^{-j\pi 1/4} = e^{-j\pi}e^{-j\pi 1/4} = -e^{-j\pi 1/4}.$$

because $e^{j\pi} = -1$. Thus, each of the first four terms in the sum is canceled by one of the last four terms.

c) If you used the alternative technique in (b), then you have already observed that $H(\omega) = 0$ when $\omega = \pi/4$.

3) **32 points**. Suppose that the frequency response H of a discrete-time LTI system *Filter* is given by: for all $\omega \in$ *Reals*,

$$H(\omega) = \cos(2\omega)$$

where ω has units of radians/sample. Give simple expressions for the output y when the input signal x : *Ints* → *Reals* is such that for all $n \in$ *Ints*, is each of the following is true:

a) $x(n) = \begin{cases} +1 & n \text{ even} \\ -1 & n \text{ odd} \end{cases}$

b) $x(n) = 5$

c) $x(n) = \cos(\pi n/2)$

d) $x(n) = \cos(\pi n/4)$

solution to problem 3:

a) This input is sinusoidal with frequency π radians per second. Since $H(\pi) = H(-\pi) = 1$, then $y(n) = x(n)$.

b) This input is sinusoidal with frequency 0, and $H(0) = 1$, so again $y(n) = x(n)$.

c) Here the input frequency is π/2. Since $H(\pi/2) = H(-\pi/2) = -1$, then $y(n) = -x(n)$.

d) Here the input frequency is π/2. Since $H(\pi/4) = H(-\pi/4) = 0$, then $y(n) = 0$.

4) **20 points** Let u be a discrete-time signal given by: for all $n \in$ *Ints*,

$$u(n) = \begin{cases} 1 & 0 \le n \\ 0 & \text{otherwise} \end{cases}.$$

This is called the **unit step** signal. Suppose that a discrete-time system H that is known to be LTI is such that if the input is u, the output is $y = H(u)$ given by: for all $n \in$ *Ints*,

$$y(n) = n\, u(n).$$

a) Find a simple expression for the output $w = H(p)$ when the input is p given by: for all $n \in$ *Ints*,

$$p(n) = \begin{cases} 2 & 0 \le n < 8 \\ 0 & \text{otherwise} \end{cases}.$$

Sketch w.

b) Find a simple expression for the impulse response h of H. Give a sketch of h.

solution to problem 4:

a) Note that $p(n) = 2(u(n) - u(n-8))$. Thus, if Δ_8 is the 8-sample delay system,

$$p = 2(u - \Delta_8(u))$$

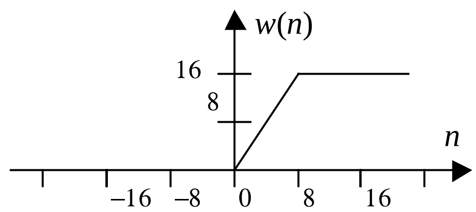
By linearity and time invariance, it must be true that

$$w = 2(y - \Delta_8(y)).$$

Thus, for all $n \in Ints$,

$$\begin{aligned} w(n) &= 2(y(n) - y(n-8)) \\ &= 2(nu(n) - (n-8)u(n-8)) \\ &= \begin{cases} 2n & 0 \le n < 8 \\ 16 & 8 \le n \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Here is a sketch (actually, this sketch should be discrete…)



c) Observe that $\delta(n) = u(n) - u(n-1)$. Thus $h(n) = nu(n) - (n-1)u(n-1) = u(n-1)$. Sketch: