EECS 20. Solution to Midterm No. 1, October 13, 1999.

1. **20 points** Fill in the blanks:

   (a) If $A = \{1, 2, 3\}, B = \{2, 3, *, \#\}$, then $A \cap B = \boxed{\{2, 3\}}$ and $A \cup B = \boxed{\{1, 2, 3, *, \#\}}$.

   (b) If the predicates P, Q, R all evaluate to $false$, then $[\neg P \wedge Q] \vee [\neg Q \wedge R] \vee [\neg R \wedge P]$ evaluates to $\boxed{false}$.

   (c) If $f : X \to Y$ and $g : Y \to Z$, then $g \circ f : \boxed{X \to Z}$.

   (d) Euler's formula is $\exp i\theta = \boxed{\cos(\theta) + i \sin(\theta)}$.

   (e) If $A \cos(\omega t + \theta) = \cos(\omega t + \pi/4) + \cos(\omega t - \pi/4)$, then $\boxed{A = \sqrt{2}, \theta = 0}$.

2. **20 points** Determine which of the following functions are periodic and what is their period in seconds or samples.

   (a) $\forall n \in Ints, \quad x(n) = \cos(2\pi n/111).$

   $\boxed{\text{Periodic, with period 111 samples}}$.

   (b) $\forall n \in Ints, \quad x(n) = \cos(2\pi\sqrt{2}n).$

   $\boxed{\text{Not periodic}}$.

   (c) $\forall t \in Reals, \quad x(t) = \cos(2\pi\sqrt{2}t).$

   $\boxed{\text{Periodic with period } 1/\sqrt{2} \text{ sec}}$.

   (d) $\forall t \in Reals, \quad x(t) = \exp(2\pi 60t + \pi/4).$

   $\boxed{\text{Periodic with period 1/60 sec}}$.

3. **20 points** Consider a discrete-time LTI system

$$H : [Ints \to Comps] \to [Ints \to Comps]$$

such that for input signal x, the output signal y is:

$$\forall\, n \in Ints, \quad y(n) = x(n) + x(n-1).$$

(a) When the input signal x is:

$$x(n) = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases}$$

the output signal y is

$$\boxed{y(n) = \begin{cases} 0, & n < 0 \\ 1, & n = 0 \\ 2, & n > 0 \end{cases}}$$

(b) Obtain an expression for the the frequency response $\hat{H}(\omega)$.
Suppose $\forall n, x(n) = e^{i\omega n}$, then

$$\begin{aligned} y(n) &= e^{i\omega n} + e^{i\omega(n-1)} = [1 + e^{-i\omega}]e^{i\omega n} \\ &= \hat{H}(\omega)e^{i\omega n} \end{aligned}$$

so

$$\boxed{\hat{H}(\omega) = 1 + e^{-i\omega} = 1 + \cos(\omega) - i\sin(\omega)}$$

(c) Expressions for the magnitude response $|\hat{H}(\omega)|$ and the phase response $\angle\hat{H}(\omega)$ for $-\pi < \omega < \pi$, can be derived as follows. We have,

$$\boxed{|\hat{H}(\omega)| = \sqrt{(1 + \cos(\omega))^2 + (\sin(\omega))^2} = \sqrt{2 + 2\cos(\omega)}}$$

and

$$\boxed{\angle\hat{H}(\omega) = -\tan^-1(\tfrac{\sin(\omega)}{1+\cos(\omega)})}$$

(d) Since $\hat{H}$ is periodic with period 2π, and since $\hat{H}(-\omega) = (\hat{H}(\omega))^*$, we only need to plot the frequency response for $0 \leq \omega \leq \pi$. Here are the plots. In drawing the plots we can use the following:

$$|\hat{H}(0)| = 2, |\hat{H}(\pi/2)| = \sqrt{2}, |\hat{H}(\pi)| = 0$$

and

$$\angle\hat{H}(0) = 0, \angle\hat{H}(\pi/2) = -\pi/4, \angle\hat{H}(\pi) = -\pi/2.$$

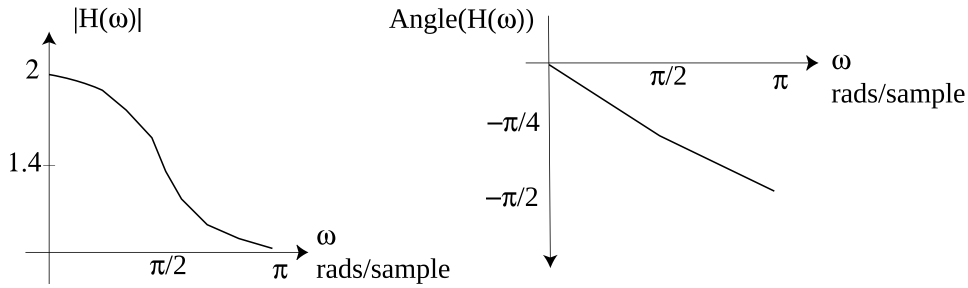

Figure 1: Frequency response $\hat{H}$

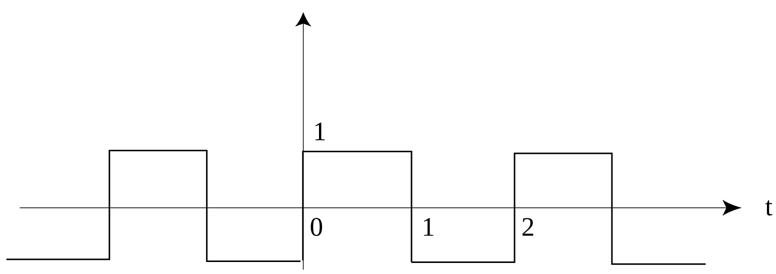

Figure 2: Square wave with period 2 seconds

4. The exponential Fourier series of the square wave periodic function x depicted in the figure is of the form:

$$\forall t \in Reals, \quad x(t) = \sum_{k=-\infty}^{\infty} X_k \exp(ik\omega_0 t). \tag{1}$$

   (a) What is ω_0? $\omega_0 = 2\pi/2 = \pi$ rads/s.

   (b) Calculate the coefficients X_k in (1).

The general formula is

$$\begin{aligned} X_k &= \frac{1}{2}\int_0^2 x(t)e^{-ik\omega_0 t}dt \\ &= \frac{1}{2}\int_0^2 x(t)e^{-ik\pi t}dt \\ &= \frac{1}{2}[\int_0^1 x(t)e^{-ik\pi t}dt - \int_1^2 x(t)e^{-ik\pi t}dt] \end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{2\pi ki}\{e^{-ik\pi t}|_{t=0}^{t=1} - e^{-ik\pi t}|_{t=1}^{t=2}\} \\
&= -\frac{1}{2\pi ki}\{2e^{-ik\pi} - 1 - e^{-2ik\pi}\}
\end{aligned}
$$

Since $e^{-ik\pi} = 1$ or -1, according as k is even or odd, whereas $e^{-2ik\pi} = 1$ for all k, this simplifies to:

$$
X_k = \begin{cases} \frac{2}{ik\pi}, & \text{if } k \text{ is odd} \\ 0, & \text{if } k \text{ is even} \end{cases}
$$