EECS 20. Midterm No. 2 Solution
April 5, 2000.

1. (a) The fundamental frequency is $\omega_0 = 1$. The Fourier series coefficients are $A_0 = A_1 = A_2 = 1$ and $\phi_1 = -\pi/2$, with everything else having value 0.

   (b) The output will have Fourier series coefficients A_k scaled by the frequency response $H(\omega)$, so $A_0 = 1$ and $A_1 = 1/2$, and all others are 0. The phases of the frequency response add to those of the input, so the output will have $\phi_1 = -\pi/2 + \pi/2 = 0$. I.e., $\phi_k = 0$ for all k. Thus, the output is

   $$y(t) = 1 + \cos(t)/2.$$

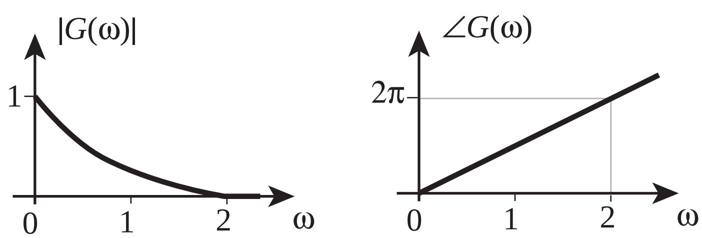
   (c) The frequency, magnitude and phase responses of the cascade composition are

   $$G(\omega) = H^2(\omega),$$

   $$|G(\omega)| = |H(\omega)|^2,$$

   $$\angle G(\omega) = 2\angle H(\omega).$$

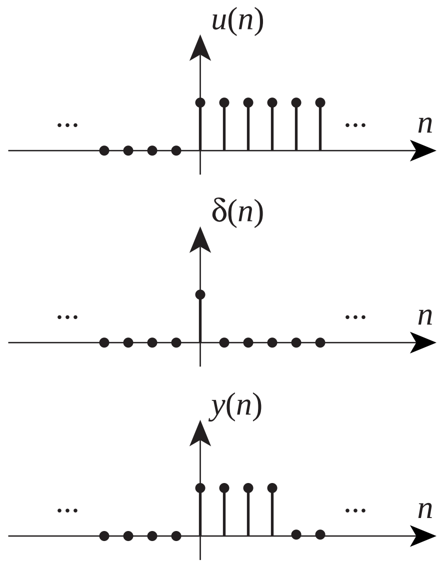
   Here is a sketch:



   (d) The Fourier series coefficients of the input will now be scaled by $G(\omega)$ instead of $H(\omega)$, getting $A_0 = 1$ $A_1 = 1/4$, and $\phi_1 = -\pi/2 + \pi = \pi/2$. Thus, the output is

   $$y(t) = 1 + \cos(t + \pi/2)/4 = 1 - \sin(t)/4.$$

2. (a) The sketches are shown below:



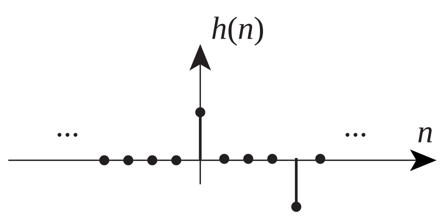
(b) Note that

$$\delta(n) = u(n) - u(n-1).$$

Since the system is LTI, it must therefore be true that

$$h(n) = y(n) - y(n-1) = \delta(n) - \delta(n-4).$$

This is sketched below:



3. (a) Note that

$$\begin{aligned}\cos^2(\pi t/6) &= (e^{i\pi t/6} + e^{-i\pi t/6})^2/4 \\ &= (e^{i2\pi t/6} + 2 + e^{-i2\pi t/6})/4 \\ &= (1 + \cos(2\pi t/6))/2.\end{aligned}$$

Moreover,

$$\sin(\pi t/6) = \cos(\pi t/6 - \pi/2).$$

Therefore

$$x(t) = 0.5 + \cos(\pi t/6 - \pi/2) + 0.5\cos(2\pi t/6).$$

(b) Using the results of part (a), $\omega_0 = \pi/6$; $A_0 = 0.5$, $A_1 = 1$, $A_2 = 0.5$, and $A_k = 0$ for $k > 2$; and $\phi_1 = -\pi/2$, and $\phi_k = 0$ for $k > 1$.

(c) Rewriting the result from part (a),

$$\begin{aligned} x(t) &= 0.5 + \cos(\pi t/6 - \pi/2) + 0.5\cos(2\pi t/6) \\ &= 0.5 + 0.5e^{i(\pi t/6 - \pi/2)} + 0.5e^{-i(\pi t/6-\pi/2)} + 0.25e^{i2\pi t/6} + 0.25e^{-i2\pi t/6}. \end{aligned}$$

From this, we can read of the Fourier series coefficients, $X_0 = 0.5$, $X_1 = 0.5e^{-i\pi/2} = -j/2$, $X_{-1} = 0.5e^{i\pi/2} = j/2$, $X_2 = X_{-2} = 0.25$, and $X_k = 0$ for $k > 2$ or $k < -2$.

4. (a) Note from the difference equation that what we need to remember about the past is $y(n-1)$. Thus, define the state to be

$$s(n) = y(n-1).$$

(You could equally well choose to define the state to be $-0.9y(n-1)$, among other possible choices.) Thus, this is a one-dimensional SISO system. The state update equation becomes

$$s(n+1) = -0.9s(n) + x(n)$$

because $s(n+1) = y(n)$. Thus, $A = -0.9$, a 1×1 matrix, and $b = 1$. The output equation is

$$y(n) = -0.9s(n) + x(n)$$

from which we recognize $c = -0.9$ and $d = 1$.

(b) Let the input $x = \delta$, the Kronecker delta function, and note that

$$y(n) = \begin{cases} 0, & n < 0 \\ 1, & n = 0 \\ -0.9 & n = 1 \\ 0.81 & n = 2 \\ (-0.9)^n & n > 2 \end{cases}$$

This can be written more compactly as $y(n) = (-0.9)^n u(n)$, where $u(n)$ is the unit step function.