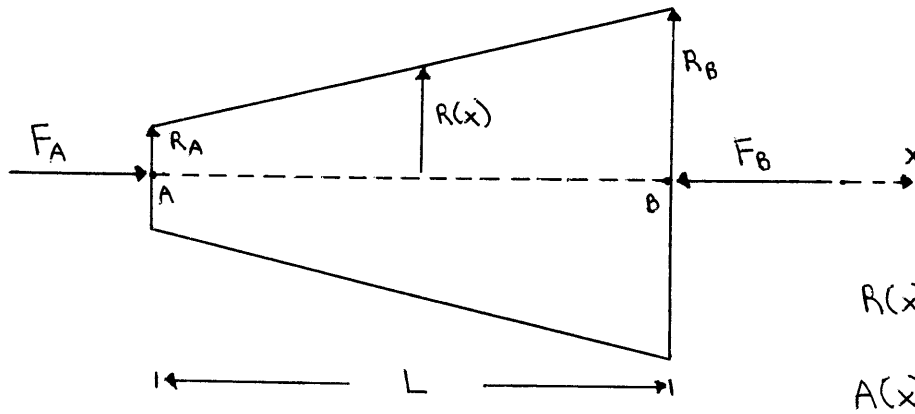
Midterm exam 2 solutions

Problem 1

a.



$$R(x) = R_A + \left(\frac{R_B - R_A}{L}\right)x$$

$$A(x) = \pi[R(x)]^2$$

equilibrium: $F_A - F_B = 0 \longrightarrow F_A = F_B = F$

compatibility: $\delta_{B\backslash A} = 0$ where $\delta_{B\backslash A} = \int_0^L \frac{-F}{EA(x)}dx + \alpha\Delta TL$

$$-\frac{F}{E\pi}\int_0^L \frac{dx}{\left[R_A + \left(\frac{R_B-R_A}{L}\right)x\right]^2} + \alpha\Delta TL = 0$$

or,

$$F = \frac{\pi E\alpha\Delta TL}{\int_0^L \left[R_A + \left(\frac{R_B-R_A}{L}\right)x\right]^{-2}dx}$$

let $u = R_A + \left(\frac{R_B-R_A}{L}\right)x$

then $du = \left(\frac{R_B-R_A}{L}\right)dx$ and $u(0) = R_A$, $u(L) = R_B$

so,

$$\int_0^L \left[R_A + \left(\frac{R_B-R_A}{L}\right)x\right]^{-2}dx = \int_{R_A}^{R_B}\left(\frac{L}{R_B-R_A}\right)\frac{1}{u^2}du$$

$$= \left(\frac{L}{R_B-R_A}\right)\cdot\left(-\frac{1}{u}\right)\Bigg|_{R_A}^{R_B}$$

$$= \frac{L}{R_A R_B}$$

Problem 1

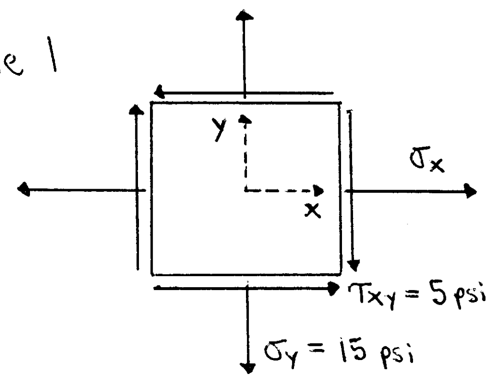
a. $\longrightarrow$ $$\boxed{F = \pi E \alpha \Delta T R_A R_B}$$

b. $\sigma_{AVG}(x) = \dfrac{F}{A(x)}$ so $(\sigma_{AVG})_{MAX} = \dfrac{F}{A_{MIN}} = \dfrac{F}{A(0)} = \dfrac{F}{\pi R_A^2}$

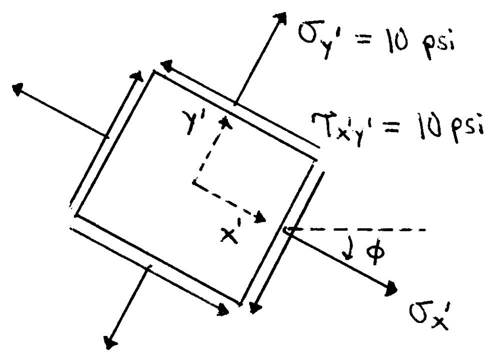
$\longrightarrow$ $$\boxed{(\sigma_{AVG})_{MAX} = E\alpha\Delta T \left(\frac{R_B}{R_A}\right) \text{ at } A\ (x=0)}$$

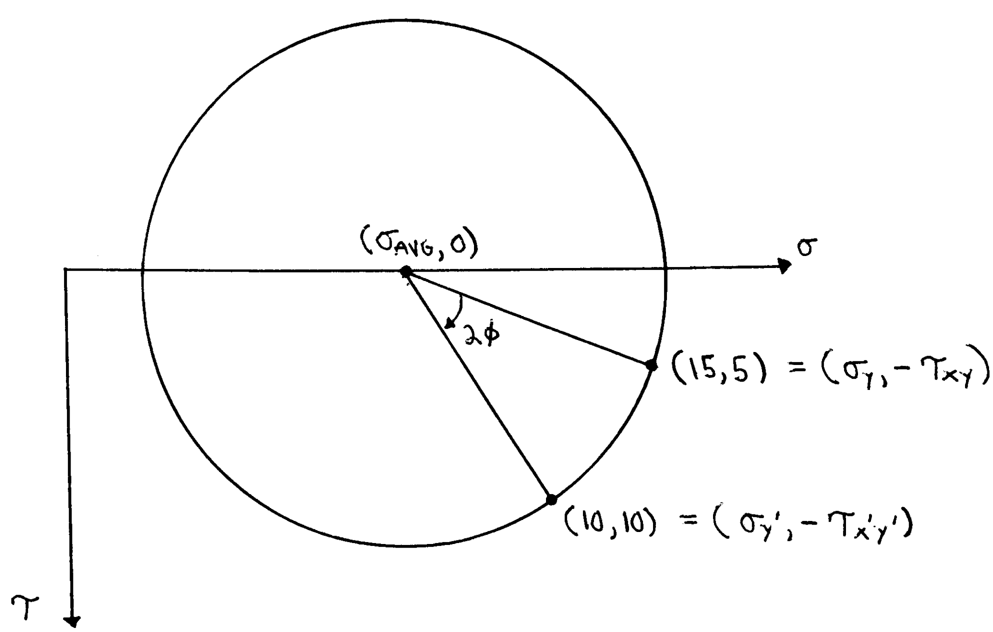
Problem 2

a. plane 1



plane 2





Problem 2

a.

equate radii :

$$(15-\sigma_{AVG})^2 + 5^2 = (10-\sigma_{AVG})^2 + 10^2$$

$$225 - 30\sigma_{AVG} + \sigma_{AVG}^2 + 25 = 100 - 20\sigma_{AVG} + \sigma_{AVG}^2 + 100$$

$$50 = 10\,\sigma_{AVG}$$

$$\sigma_{AVG} = 5 \text{ psi}$$

angle :

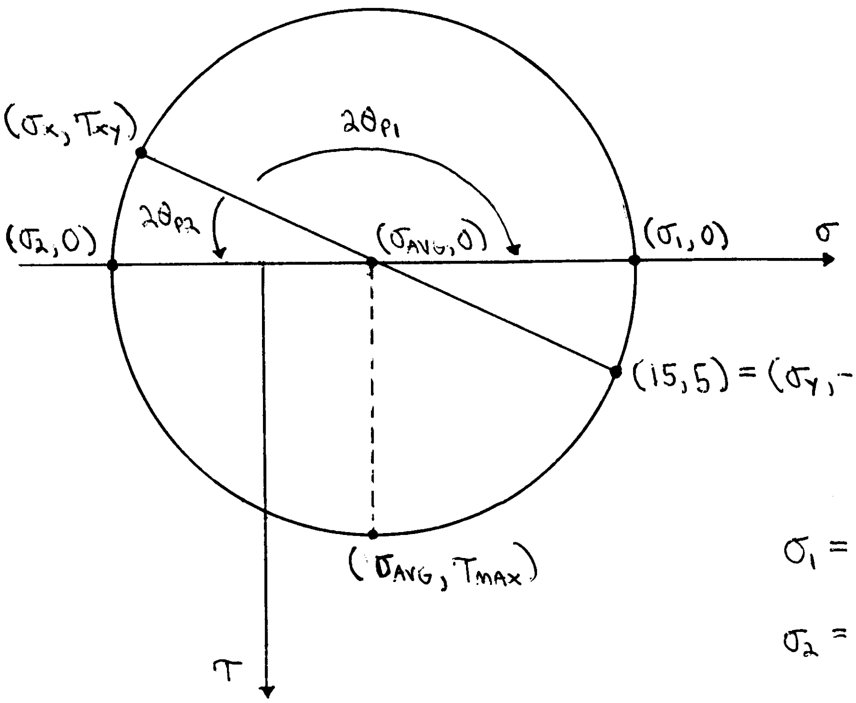
$$2\phi = \tan^{-1}\left(\frac{10}{10-\sigma_{AVG}}\right) - \tan^{-1}\left(\frac{5}{15-\sigma_{AVG}}\right)$$

$$\boxed{\phi = 18.4^\circ}$$

b.

$$\sigma_{AVG} = 5 \text{ psi}$$

$$\tau_{MAX} = R = \sqrt{(10-\sigma_{AVG})^2 + 10^2} = 5\sqrt{5} = 11.2 \text{ psi}$$



$$\sigma_1 = \sigma_{AVG} + \tau_{MAX} = \boxed{16.2 \text{ psi}}$$

$$\sigma_2 = \sigma_{AVG} - \tau_{MAX} = \boxed{-6.2 \text{ psi}}$$

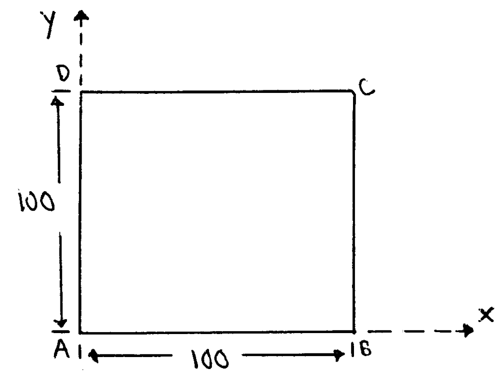
$$\theta_{p2} = \frac{1}{2}\tan^{-1}\left(\frac{5}{15-\sigma_{AVG}}\right) = \boxed{13.3^\circ \text{ counter-clockwise}}$$

$$\theta_{p1} = 90^\circ - \theta_{p2} = \boxed{76.7^\circ \text{ clockwise}}$$

c.

from (b), $\boxed{\tau_{MAX} = 11.2 \text{ psi}}$ and $\boxed{\sigma_{AVG} = 5 \text{ psi}}$

Problem 3



$$u(x,y) = 0.01x + 0.05y$$

$$v(x,y) = -0.03x + 0.02y$$

a.

$$\varepsilon_x = \frac{\partial u}{\partial x} = \boxed{0.01}$$

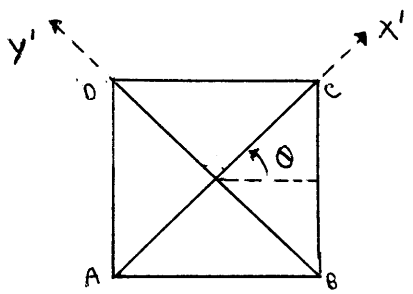
$$\varepsilon_y = \frac{\partial v}{\partial y} = \boxed{0.02}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0.05 - 0.03 = \boxed{0.02}$$

b.

$$\overline{AB} = (1+\varepsilon_x)(\overline{AB_0}) = 1.01(100) = \boxed{101}$$

c.



$$\theta = 45°$$

$$\gamma_{x'y'} = (\varepsilon_y - \varepsilon_x)\sin 2\theta + \gamma_{xy}\cos 2\theta$$

$$= \varepsilon_y - \varepsilon_x$$

$$= 0.01$$

$$\theta_{AC,BD} = \frac{\pi}{2} - \gamma_{x'y'} = 1.56 \text{ rad} = \boxed{89.4°}$$

d.

$$E = 100 \text{ GPa}$$

$$\nu = 0.3$$

$$G = \frac{E}{2(1+\nu)} = \frac{500}{13} \text{ GPa} \approx 38.5 \text{ GPa}$$

Problem 3

d. $\tau_{xy} = G\gamma_{xy} = \left(\frac{500}{13}\,GPa\right)(0.02) = 769\ MPa$

stress-strain relations:
$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right] \quad (1)$$
$$\varepsilon_y = \frac{1}{E}\left[\sigma_y - \nu(\sigma_x + \sigma_z)\right] \quad (2)$$
$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - \nu(\sigma_x + \sigma_y)\right] \quad (3)$$

plane strain: $\varepsilon_z = 0$ so $(3) \longrightarrow \sigma_z = \nu(\sigma_x + \sigma_y)$

substituting in (1) and (2),

$$E\varepsilon_x = \sigma_x - \nu\left[\sigma_y + \nu(\sigma_x + \sigma_y)\right] = (1-\nu^2)\sigma_x - \nu(1+\nu)\sigma_y$$
$$E\varepsilon_y = \sigma_y - \nu\left[\sigma_x + \nu(\sigma_x + \sigma_y)\right] = -\nu(1+\nu)\sigma_x + (1-\nu^2)\sigma_y$$

or,

$$1\ GPa = 0.91\,\sigma_x - 0.39\,\sigma_y$$
$$2\ GPa = -0.39\,\sigma_x + 0.91\,\sigma_y$$

so,

$$\begin{aligned} 1.0989 &= \sigma_x - 0.4286\,\sigma_y \\ + \quad 5.1282 &= -\sigma_x + 2.3333\,\sigma_y \\ \hline 6.2271 &= 1.9047\,\sigma_y \end{aligned}$$

$\longrightarrow$
$$\boxed{\begin{aligned} \sigma_y &= 3.27\ GPa \\ \sigma_x &= 2.50\ GPa \\ \sigma_z &= \nu(\sigma_x + \sigma_y) = 1.73\ GPa \end{aligned}}$$