Midterm 2 Solutions
EE126 - Fall 2000

Problem 1.

$$X, Y \sim unif(-1, 1)$$

$$Z = XY$$

Let $U = X$ and $V = XY$ and form the jacobian for $f_{UV}(u, v)$.

$$f_{UV} = \frac{1}{|U|} f_{XY}(u, v/u)$$

Marginalize with respect to u and employ the independence of X and Y.

$$f_V(v) = f_Z(z) = \int_{-\infty}^{+\infty} \frac{1}{|U|} f_X(u) f_Y(z/u) du$$

$$f_Z(z) = \int_{-\infty}^{+\infty} \frac{1}{2|U|} f_Y(z/u) du$$

The domain for f_Y is bounded away from 0 by the z/u term.

First consider $z > 0$ for $f_Y(z/u) \neq 0$, $1 > |u| > z$. Therefore, the integral above can be completed for $z > 0$ by splitting it up into when $z < u < 1$ and $-1 < u < -z$.

so, for $z > 0$

$$f_Z(z) = \int_z^1 \frac{1}{4U} du + \int_{-1}^{-z} \frac{1}{-4U} du$$

$$f_Z(z) = \int_z^1 \frac{1}{2U} du$$

$$f_Z(z) = -\frac{1}{2} ln(z)$$

now, similarly, for $z < 0$

$$f_Z(z) = \int_{-1}^{-z} \frac{1}{4U} du + \int_z^1 \frac{1}{-4U} du$$

$$f_Z(z) = \int_z^1 \frac{1}{-2U} du$$

$$f_Z(z) = -\frac{1}{2} ln(-z)$$

so

$$f_Z(z) = -\frac{1}{2} ln(|z|)$$

it is undefined for z=0, but that's ok, because it happens with a probability of 0. The integral can be evaluated in a closed form though.

b)

since X is independent of Y

$$E[XY] = E[X]E[Y] = 0$$

Problem 2.

$$P(T > t) = \frac{1}{1+t}$$

$$F(t) = \frac{t}{1+t}$$

$$f_T(t) = \frac{dF_T}{dt} = \frac{1}{(1+t)^2}$$

b.

$$P(atleast1bulbworkingatt = 9givenworkingatt = 1)$$

$$= 1 - P(0workingatt = 9|4workingatt = 1)$$

$$= \frac{P(0att = 9, 4att = 1)}{P(4att = 1)}$$

$$P(t < 9, t > 1) = \frac{t_2}{t_2 + 1} - \frac{1}{1 + t_1} = \frac{2}{5}$$

$$P(t > 1) = \frac{1}{2}$$

$$P(0workingatt = 9|allworkingatt = 1) = \frac{4}{5}^4$$

$$P(atleast1bulb \ldots) = 1 - \frac{4}{5}^4$$

c.

$$Z = min(T_1, T_2, T_3, T_4)$$

$$P(Z = t) = \binom{4}{1} P(T_1 = t) P(T_2 > t) P(T_3 > t) P(T_3 > t)$$

$$f_T(t) = \frac{4}{(1+t)^5}$$

d.
Consider 1 case:

$$f_Z(t) = P(T_1 = t)P(T_2 < t)P(T_3 < t)P(T_4 > t)$$

there are 3 combinations that have $T_1 = t$ and 4 positions for $T_X = t$ so...

$$f_Z(t) = 12\frac{1}{(1+t)^2}\left(\frac{t}{1+t}\right)^2 \frac{1}{1+t}$$

$$= \frac{12t^2}{(1+t)^5}$$

Problem 3.

Consider $X+Y$ Since this is also a gaussian distribution. It is appropriate to find if it is independent of $X-Y$.

$$E[(X-Y)(X+Y)] = E[X^2] - E[Y^2] = 1 - 1 = 0$$

and since

$$f(X-Y) \amalg g(X+Y)$$

$$E\left[(X+Y)^4 | X-Y\right] = E\left[(X+Y)^4\right]$$

After expansion and noting that E[X] = E[Y] = 0 and X and Y are independent...

$$= E[X^4] + E[Y^4] + 6E[X^2Y^2] = 3 + 3 + 6 = 12$$

Problem 4.

$$r = \sqrt{x^2+y^2}, \phi = \arctan\left(\frac{y}{x}\right)$$

$$x = r\cos(\phi), y = r\sin(\phi)$$

forming the inverse jacobian...

$$J = \begin{bmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{bmatrix}$$

Taking the determinat gives $J^{-1} = r$

$$f_{r\phi}(r, \phi) = r \cdot f_{XY}(r\cos(\phi), r\sin(\phi))$$

Problem 5. extra credit

The characteristic function of a Poission rv is...

$$G(s) = \exp(\lambda(s-1))$$

The sum of two independent rv's is the product of their characteristic functions...

$$\begin{aligned} H(s) &= \exp(a(s-1)) \cdot \exp(b(s-1) \\ &= \exp((a+b)(s-1)) \end{aligned}$$

so, converting back the characteristic function, which is in standard form...

$$f_Z(k) = \frac{(a+b)^k}{k!} \exp(-(a+b)k)$$